NEAR RINGS OF QUOTIENTS

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TO MY PARENTS

CERTIFICATE

Certified that the work in this dissertation entitled "NEAR RINGS OF QUOTIENTS" by V. Seth has been carried out under my supervision and that this work has not been submitted elsewhere for any degree.

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INTRODUCTION

The concept of Near-Rings arises very naturally, if we define addition and multiplication on the set N of all mappings (which preserve additive identity) of an additive (not necessarily abelian) group G into itself. The addition and multiplication of two mappings f,h is given by $(f+h)(x) = f(x) + h(x)$ $(fh)(x) = f(h(x))$, for all x in G. Then $(N, +)$ is a group (not necessarily abelian). Also only one distributive law holds. So that $(N, +.)$ is a near-ring, which is not a ring in general.

Zassenhaus [19] in 1936 studied the finite near-fields. The structure theorems on simple and semi-simple near-rings were first proved by Blackett [5] in 1953. Later Frohlich and Laxton studied the d.g. near rings; and Betsch, Beidleman, Maxson, Ramakotiah, Ligh Steve and others generalised various concepts from rings to near-rings.

In this thesis we have generalised certain concepts, like classical-ring of quotients, complete ring of quotients, generalised centralizers of a module etc. to near-rings and have studied some of their properties.

The present thesis consists of five chapters. Chapter I contains the known definitions and results, which are required in the ensuing chapters.

In Chapter II we give criteria for injectivity of a module over a d.g. near ring and construct an injective cover for it. We have also shown that if M is a right N (not necessarily d.g.) module which can be embedded in an injective module, then M has an injective hull. Hence any module over a d.g. near ring has an injective hull. Some equivalent conditions are given for a module to be an injective hull of a given near-ring module.

Maxson [15] gave a necessary and sufficient condition for the existence of a near ring of left quotients of a left near ring with 1. He also gave a sufficient condition for the existence of a near ring of right quotients of a left near ring with 1.

What Maxson calls a near ring of quotients, we call it a classical near ring of quotients. In Chapter III we give another construction (using Asano's method) of classical near rings of left and right quotients and give examples to show that the concepts of classical left near ring of left quotients and classical left near ring of right quotients are quite independent of each other.

Analogous to the construction of the classical near ring Q' of right quotients, we give the construction of a module $\overline{M}$ of quotients for any right module M over a d.g. near-ring N with 1.

For any S-free module M we give a necessary and sufficient condition for $\overline{M}$ to be injective. As a Corollary to this result we

get a necessary and sufficient condition for Q'_N to be injective.

Chapter IV is devoted to the study of complete near rings of left and right quotients.

For any left d.g. near ring N with 1 we define
$H = \{f: I \to I \mid (is)f = (i)f\,s$ for all i in I and all distributive s in N$\}$ (where I_N denotes the injective hull of N_N) and
$Q = \{\theta : I \to I \mid (h.i)\theta = ((i)h)\theta = ((i)\theta)h$ for all i in I and all N-homomorphism h in H$\}$,

and show that Q is a left near ring of right quotients of N and that it contains any left near ring of right quotients of N. It is also shown that Q is its own near ring of right quotients. Q is called the complete near ring of right quotients of N.

It is proved that Q is a regular near ring whenever $J(N_N) = 0$. Moreover it is shown that if every weakly-large right N-subgroup of N contains a distributive non-zero divisor then Q is a classical near ring of right quotients of N.

Moreover given any left near ring N with 1, we construct a left near ring C containing N. We show that C is a left near ring of left quotients of N and it contains any left near ring of left quotients of N. It is also shown that C is its own near ring of left quotients. C is called the complete near ring of left quotients of N.

In the last chapter we have constructed generalised centralizer of a near-ring module. For this we have taken a collection $\tau'(\tau)$ of N-subgroups (N-submodules) of a module M (over a d.g. near ring N) satisfying certain conditions, and with each such collection $\tau'(\tau)$ we have associated a near ring $P'(P)$. The near ring P' turns out to be non-associative.

In case N is a ring and τ' is the collection of large submodules of a given module, then the near ring P' turns out to be an (associative) ring and this ring is actually Johnson's extended centralizer of a module [11].

It is also proved that if the near-ring module M is τ^*-complemented (τ - complemented) in a certain sense then the near ring $P'(P)$ is regular.

We now explain the method for referring various results. Throughout any one chapter the results have been indicated in International numerals. Thus 4.5 refers to the result 4.5 of Chapter IV. When reference is to some result of another chapter the number of the chapter appears first in Roman numerals. Thus III.4.5 refers to the result 4.5 of Chapter III.

CHAPTER I

PRELIMINARIES

In this chapter, we give all definitions and results which are needed in this thesis.

1. NEAR RING, N-MODULE, DEFINITIONS, EXAMPLES.

1.1. A left near_ring N is a non empty set with two binary operations: addition (+) and multiplication (.), satisfying the following conditions:

(a) $(N, +)$ is a group (not necessarily abelian)

(b) $(N, .)$ is a semi group

(c) $z(x + y) = zx + zy$ for every $x, y, z \in N$

(d) $0.r = 0$ for every $r \in N$.

A right near ring satisfies the right distributive law instead of the left. The difference between near-rings and rings is that for near-rings addition need not be commutative and only one distributive law holds.

From now on the term 'near-ring' shall mean left near-ring.

1.2. Examples. (a) Let G be a nontrivial additive group (non-abelian) and let $T_0(G)$ denote the set of all maps from G to G such that the zero of G is mapped to itself. $T_0(G)$ becomes a near-ring (which is not a ring) under the addition and multiplication of

maps defined in the following way :

for $f, h \in T_0(G)$, $x \in G$

$$(x)(f + h) = (x)f + (x)h$$

$$(x)fh = ((x) f)h$$

(b) Let G be an arbitrary additive group (not necessarily abelian). Then for every $y \in G$, define

$$yx = x,$$

and $$0x = 0, \quad \text{for every } x \in G.$$

Then $(G, +, .)$ is a near-ring.

This example shows that every group can be made into a near-ring.

1.3. The following properties are immediate for a near-ring

(a) $r.0 = 0$ for every $r \in N$,

and (b) $-(xy) = x(-y)$ for every $x, y \in N$.

1.4. If N contains an element $1 \neq 0$ such that

$r.1 = 1.r = r$ for every $r \in N$, then 1 is called identity for N. In this case N is said to be unitary near ring or a near-ring with identity, also written as '$1 \in N$'.

1.5.(a) An element $y \in N$ is said to be (right) distributive if $(x + z)y = xy + zy$ for every $x, z \in N$.

(b) A near-ring N is said to be distributively generated (d.g.) [8,13] if there exists a multiplicative semigroup S of distributive

elements of N such that the additive group (N, +) is generated by the set S. In this case S is called a generating set for N.

If N is a d.g. near-ring with a generating set S, then every element $x \in N$ can be written in the form

$$x = \sum_{i=1}^{k} \pm s_i \text{ (for some k)}, \ s_i \in S, \text{ (see [8], [13])}$$

Also the axiom $0.r = 0$ ($r \in N$) is always satisfied in d.g. near rings ([8], [15])

Remark : N is a d.g. near-ring iff (N, +) is generated by the semi-group of all the distributive elements of N. Hence without loss of generality we may take the elements 0 and 1 of N in the generating set of a d.g. near ring N.

1.6. Example [2] . Let E be a multiplicative semigroup of endomorphisms of a nontrivial group (G, +) and let E(G) be the subgroup of the additive group $T_0(G)$ (example 1.2(a)) generated by E, then E(G) is a d.g. near-ring with identity.

Analogous to the ring module concept, a near-ring module is defined as follows:

1.7. (a) A right N-module M(or M_N) is a system (M, +,.) such that

(i) (M, +) is a group (not necessarily abelian).

(ii) '.' is a map from $M \times N \to M$, given by $m.r = mr \in M$ and satisfying the following conditions:

(a') $m(r_1 + r_2) = mr_1 + mr_2$,

(b') $m(r_1 \cdot r_2) = (mr_1)\, r_2$,

for every $r_1, r_2 \varepsilon N$, $m \varepsilon M$.

If N is a d.g. near-ring with a generating set S, then M_N should satisfy one more condition :

(c') $(m_1+m_2)s = m_1 s + m_2 s$, for every $s \varepsilon S$, $m_1, m_2 \varepsilon M$.

(b) A left N module M(or $_N M$) is a system (M, +, .) such that

(i) (M, +) is a group (not necessarily abelian)

(ii) '.' is a map from $N \times M \to M$, given by $r.m = rm \varepsilon M$

and satisfying the following conditions :

(a") $n(m_1+m_2) = nm_1 + nm_2$,

(b") $(n_1 . n_2)m = n_1(n_2 m)$,

for every $n, n_1, n_2 \varepsilon N$, $m, m_1, m_2 \varepsilon M$.

If N is a near-ring with identity 1 and $m.1 = m$ $(1.m = m)$ for every $m \varepsilon M$, then $M_N(_N M)$ is said to be unitary N-module.

1.8. Examples. (a) Every near-ring N is a left as well as a right N-module.

(b) Any nontrivial group (G, +) is unitary $T_o(G)$-module as well as E(G)-module, ([2]).

1.9. Some different names for N-module have appeared in the literature. Betsch, Fröhlich, Laxton and Ramakotiah have called N-module as 'N-group' and Blackett has called it as N-space.

We now define the important concepts of 'N-subgroup' and 'N-submodule'.

1.10. Let M_N $({}_N M)$ be an N-module and A be a non-empty subset of M. Then A is said to be right (left) N-subgroup of M, if

(a) $(A, +)$ is a subgroup of $(M, +)$,

and (b) $AN = \{ar \mid a \in A,\ r \in N\} \subseteq A$ $(NA = \{ra \mid a \in A,\ r \in N\} \subseteq A)$.

When $M = N$ and A is as above then we say that A is an N-subgroup.

1.11. A non-empty subset B of M is called a right N-submodule of M if

(a) $(B, +)$ is a normal subgroup of $(M, +)$,

and (b) $(m+b)r-mr$ (or equivalently $(b+m)r-mr$) $\in B$, for every $r \in N$, $m \in M$, $b \in B$.

If N is a d.g. near-ring, then B is an N-submodule of M iff B is a normal N-subgroup of M, ([13]).

N-submodules are called normal N-modules by Fröhlich and Laxton.

1.12. If N happens to be a ring, then the concepts of N-subgroup and N-submodule coincide. But in near-rings every N-submodule is an N-subgroup, while the converse does not always hold. For this we present the following example.

1.13. Example [2] . Consider the near-ring $T_o(G)$ of 1.2 and let $g \in G$. Define $\theta_g \in T_o(G)$ as follows:

$$(x)\theta_g = g \text{ for every } x(\neq 0) \text{ in } G$$
$$= 0 \text{ if } x = 0.$$

Let $K_G = \{\theta_g | g \in G\}$. Then K_G is a sub-near-ring of $T_o(G)$. It is also called the near-ring of constant mappings of the group G. If the order of $(G,+)$ is greater than two, then it has been shown in ([2] - example 1.15) that K_G is a $T_o(G)$ - subgroup of $T_o(G)$, but not a $T_o(G)$- submodule of $T_o(G)$.

1.14. We have seen in near-rings that N-subgroups may not be N-submodules. We now present an example to show that there exist near-rings and near-ring modules (other than rings and ring module) in which every N-subgroup is an N-submodule.

1.15. Example. [2] . Consider the d.g. near ring $E(G)$ of 1.6. Then it has been shown in ([2] - Lemma 3.3) that every $E(G)$-subgroup of $G_{E(G)}$ is also an $E(G)$-submodule.

1.16. Let X be a given subset of M. Then the intersection of all the N-submodules (N-subgroups) of M containing X is an N-submodule (N-subgroup) of M and is called the N-submodule (N-subgroup) generated by X.

1.17. Let M be an N-module, A be an N-subgroup of M and B be an N-submodule of M. Then A+B is an N-subgroup of M and A+B = B+A, ([2] - proposition 1.24).

1.18. Let $\{M_\alpha | \alpha \in I\}$ be a collection of N-submodules of M. Then $\sum_{\alpha \in I} M_\alpha$ is an N-submodule of M and consists of all the finite sums of the elements from $M_\alpha^{'s}$, ([2] - proposition 1.25).

§ 2. Homomorphisms, Ideals, Isomorphism Theorems.

2.1 (a) A mapping $\theta : M_N \to M'_N$ is called an N-homomorphism if

$$\theta(x+y) = \theta(x) + \theta(y)$$

and
$$\theta(xr) = \theta(x)r,$$

for every $x, y \in M$ and $r \in N$.

N-epimorphisms, N-monomorphisms and N-isomorphisms are defined in the usual way. If M and M' are N- isomorphic, then we shall write $M \underset{(N)}{\cong} M'$.

(b) A mapping $\theta : N \to N'$(N and N' being near-rings) is called a near-ring homomorphism, if it preserves both the operations.

Near-ring epimorphism, monomorphism, isomorphism are defined as usual.

If N and N' are isomorphic near-rings, then we shall write $N \cong N'$.

2.2. Let $\theta : M \to M'$ be an N-homomorphism, then $\theta(M) = \{\theta(x) | x \in M\}$ is an N-subgroup of M', but need not be an N-submodule of M'. (see [2],[15]).

For this we give the following example :

2.3 Example . Consider the near-ring K_G of 1.13, which is a $T_o(G)$-subgroup of $T_o(G)$ but not a $T_o(G)$ - submodule of $T_o(G)$. K_G is also a $T_o(G)$ - module under the following definition:

$$\theta_g f = \theta_{(g)f} \quad \text{for every } f \in T_o(G), \quad \theta_g \in K_G.$$

Now define a map $\psi : K_G \to T_o(G)$ as follows:

$$\psi(y) = y, \; y \in K_G.$$

Then ψ is $T_o(G)$ -homomorphism and $\psi(K_G) = K_G$ is a $T_o(G)$-subgroup of $T_o(G)$ but not a $T_o(G)$-submodule of $T_o(G)$.

Contrary to the ring case, we have seen that in near-ring modules, $\theta(M)$ may not be an N-submodule of M'. We therefore, have the following definition:

2.4. Let $\theta : M \to M'$ be an N-homomorphism. If $\theta(M)$ is an N-submodule of M', then θ is called a normal N-homomorphism [15].

We now give the definitions of left ideal, right ideal and ideal in near-rings.

2.5. (a) A non-empty subset B of N is called a left ideal of N if

(i) $(B,+)$ is a Normal subgroup of $(N, +)$ and

(ii) $NB = \{ rb \mid r \in N, \; b \in B \} \subseteq B$.

(b) B is called a right ideal of N if B is N-submodule of the N-module N_N.

(c) B is called (two sided) ideal of N if B is left as well as right ideal.

2.6. The right ideal (ideal) generated by a subset X of N, denoted by $\langle X \rangle_r$ ($\langle X \rangle$), is the smallest right ideal (ideal) containing the set X.

If N is d.g. near ring (with 1), then

$$<X>_r = \{ \sum_{i=1}^{k} (- r_i + x_i s_i + r_i) \mid r_i \varepsilon N,\ x_i \varepsilon X,\ s_i \varepsilon S\} \ ([13]).$$

As in ring theory, we have factor N-modules, factor near-rings, isomorphism theorems etc. in near-rings. The following results of this section are contained in [2].

2.7. (a) Let A be an N-submodule of an N-module M. Then the factor group M/A can be considered as an N-module under the following usual definition:

$$(m + A)\, r = mr + A,\ r \varepsilon N,\ m \varepsilon M.$$

The natural N-epimorphism $\pi : M \to M/A$ is given by

$$\pi(m) = m + A,\ m \varepsilon M.$$

(b) If B is an ideal of N, then the factor group N/B becomes factor-near-ring under the usual multiplication:

$$(r_1+B)\ (r_2+B) = r_1 r_2 + B,\ r_1, r_2 \varepsilon N.$$

The natural near-ring epimorphism $\pi : N \to N/B$ is given by

$$\pi(r) = r + B,\ r \varepsilon N.$$

2.8. B is an N-submodule of M(ideal of N) iff B is the Kernel of an N-homomorphism of M(near-ring homomorphism of N).

2.9. Let $\{B_\alpha\}_{\alpha\varepsilon I}$ be a collection of ideals of N, then $\sum_{\alpha\varepsilon I} B_\alpha$ is an ideal of N.

2.10. Let $\theta : M \to M'(\theta : N \to N')$ be an N-epimorphism (near-ring epimorphism), then

$$M/\mathrm{Ker}\,\theta \underset{(N)}{\cong} M'(N/\mathrm{Ker}\,\theta \cong N').$$

2.11. Let A be an N-submodule of M (ideal of N) and B be an N-submodule of M(ideal of N) containing A, then

$$\frac{M}{B} \underset{(N)}{\cong} \frac{M/A}{B/A}. \quad (N/B \cong \frac{N/A}{B/A})$$

2.12. (a) Let A be an N-subgroup of M and B be an N-submodule of M, then $(A+B)/B \underset{(N)}{\cong} A/A \cap B \underset{(N)}{\cong} \pi(A)$, where $\pi : M \to M/B$ is the natural N-epimorphism.

(b) Let A and B be ideals of N, then

$$(A+B)/B \cong A/A \cap B \quad .$$

§3. More Definitions and Results

3.1. (a) A proper N-subgroup (N-submodule) A of M is called maximal N-subgroup (maximal N-submodule) of M, if for any other N-subgroup (N-submodule) A' of M with $A \subsetneq A'$, then either A = A' or A' = M.

(b) An ideal D of N is called a maximal ideal, if for any other ideal E of N with $D \subseteq E$, either D = E or E = N.

CHAPTER II

ON INJECTIVE NEAR-RING MODULES

Analogous to the ring theory case we prove here that if N is a d.g. near-ring with 1, then for a right N-module M, the following are equivalent:

(i) M is injective

(ii) every diagram
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "j"] \arrow[d, "g"'] & D \\ & M & \end{tikzcd}$$
can be embedded into a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "j"] \arrow[d, "g"'] & D \arrow[dl, "h"] \\ & M & \end{tikzcd}$$
where C is a submodule of any right N-module D and j is the injection map.

(iii) for every right ideal u of N and every N homomorphism $f : u \to M$ there exists a m in M such that $f(a) = ma$ for all a in u.

Using this we construct an injective cover for any right N-module M. We have also shown that if M is a right N-module which can be embedded in an injective module, then M has an injective hull P for which the following are equivalent

(1) P is a maximal weakly essential extension of M.

(2) P is a weakly essential extension of M and is injective.

(3) P is a minimal injective extension of M.

§ 1. Criteria for Injectivity

Let N be a left near ring and let M be a right N-module. We recall [16] that M is called injective if and only if every diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \\ & M & \end{tikzcd}$$, where A and B are right N-modules with $0 \to A \xrightarrow{f} B$ exact, can be embedded into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

We first prove the following:

1.1. Lemma : Let N be a left d.g. near ring, with identity, generated by a distributive semi group S. Let M be a right N-module and A any subset of M. Then the submodule of M generated by A is

$$\bar{A} = \{\sum_{i=1}^{n} -m_i + a_i s_i + m_i \mid m_i \in M,\ a_i \in A \text{ and either } s_i \text{ or } -s_i \in S\}.$$

Proof : Clearly $\bar{A}$ is an N-subgroup of M.

We now show that $\bar{A}$ is normal in M.

$$-m + \{\sum_{i=1}^{n} -m_i + a_i s_i + m_i\} + m = -m + (-m_1 + a_1 s_1 + m_1) + (-m_2 + a_2 s_2 + m_2) + \ldots + (-m_n + a_n s_n + m_n) + m$$

$$= \{(-m-m_1) + a_1 s_1 + (m_1+m)\} + \ldots + \{(-m-m_n) + a_n s_n + (m_n+m)\}$$

$$= \sum_{i=1}^{n} \{(-m-m_i) + a_i s_i + \quad + (m_i+m)\} \text{ is in } \bar{A}$$

for all $m \in M$ and all $\sum_{i=1}^{n} \{-m_i + a_i s_i + m_i\}$ in $\bar{A}$. Hence $\bar{A}$ is a submodule of M. Clearly A is contained in $\bar{A}$. Suppose B is a submodule of M containing A. We claim that $\bar{A}$ is contained in B.

Let $\bar{a} = \sum_{i=1}^{n} (-m_i + a_i s_i + m_i)$ be an arbitrary element of $\bar{A}$. Since $a_i \in A \subseteq B$ and B is a submodule of M, we have $-m_i + a_i s_i + m_i$ is in B for all $i = 1, 2, \ldots n$. Hence $\bar{A} \subseteq B$.

Remark : If $(N, +)$ is generated by a distributive set and if A is a subset of M such that $AN \subseteq A$ then the submodule of M generated by A is of the form $\{ \sum_{i=1}^{n} -m_i + a_i + m_i \mid m_i \in M, a_i \in A \}$.

1.2. Theorem : Let N be a left near ring, with identity, such that $(N, +)$ is generated by a set S of distributive elements. Then the following are equivalent:

(i) M is injective

(ii) every diagram
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "j"] \arrow[d, "g"'] & D \\ & M & \end{tikzcd}$$
C being a submodule of the right-N-module D and j being the injection map, can be embedded into a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "j"] \arrow[d, "g"'] & D \arrow[dl, "h"] \\ & M & \end{tikzcd}.$$

(iii) for every right ideal u of N and every N homomorphism $f: u \to M$ there exists a m in M such that $f(a) = ma$ for all a in u.

Proof : (i) $\Rightarrow$ (ii) trivial.

(ii) $\Rightarrow$ (i) Consider
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \\ & M & \end{tikzcd}$$
where $0 \to A \xrightarrow{f} B$ is exact. Then $f(A) = C$ is a right N-module contained in B and $g': C \to M$

given by $g'(f(a)) = g(a)$ is a N-homomorphism. Let $\bar{C}$ be the submodule of B generated by C, then

$$\bar{C} = \{\sum_{i=1}^{n} -b_i + c_i + b_i \mid b_i \in B,\ c_i \in C\}.$$

Define $g'': \bar{C} \to M$ by $g''(\sum_{i=1}^{n} -b_i + c_i + b_i) = \sum_{i=1}^{n} g'(f(a_i)).(c_i = f(a_i))$

We note that $g''|_C = g'$ and $g'o\, f|_A = g$. To show that g'' is well defined we show that g'' vanishes on zero and that g'' is additive.

For this let $\sum_{i=1}^{n} (-b_i + f(a_i) + b_i) = 0$. Then

$$(-b_1+f(a_1)+b_1) = \{-b_n-f(a_n)+b_n\} + \{-b_{n-1}-f(a_{n-1})+b_{n-1}\} + \ldots + \{-b_2-f(a_2)+b_2\}$$

or $$f(a_1) = [(b_1-b_n)+ f(-a_n)+(b_n-b_1)] + [(b_1-b_{n-1})+ f(-a_{n-1})+(b_{n-1}-b_1)] + \ldots + [(b_1-b_2) + f(-a_2)+ (b_2-b_1)] \in C.$$

Hence $$g'(f(a_1)) = g'(\text{R.H.S.}) = g''(\text{R.H.S.}) = g'(f(-a_n))+ g'(f(-a_{n-1})+\ldots + g'(f(-a_2))$$

or $$g(a_1) = g(-a_n) + g(-a_{n-1})+ \ldots + g(-a_2)$$

$$= -g(a_n) - g(a_{n-1}) - \ldots - g(a_2)$$

or $$\sum_{i=1}^{n} g(a_i) = 0 = g''(\sum_{i=1}^{n} -b_i + f(a_i) + b_i)$$

Now let $\sum_{i=1}^{n} (-b_i + f(a_i) + b_i)$, $\sum_{i=1}^{m} (-b_i' + f(a_i') + b_i')$ be any two

elements of $\overline{C}$. Then $g''\left[\sum_{i=1}^{n}(-b_i + f(a_i) + b_i) + \sum_{i=1}^{m}(-b_i' + f(a_i') + b_i')\right]$

$$= \sum_{i=1}^{n} g'(f(a_i)) + \sum_{i=1}^{m} g'(f(a_i'))$$

$$= g''\left[\sum_{i=1}^{n}(-b_i + f(a_i) + b_i)\right] + g''\left[\sum_{i=1}^{m}(-b_i' + f(a_i') + b_i')\right]$$

It can be checked that g'' is N-linear and hence a N-homomorphism. Thus by (ii) there exists a N-homomorphism $h: B \to M$ such that $h|_{\overline{C}} = g''$. We claim that $h \circ f = g$. For this let $a \in A$, then $f(a)$ is in C and hence in $\overline{C}$.

So that $h(f(a)) = g''(f(a)) = g'(f(a)) = (g' \circ f)(a)$ for all a in A

or $(h \circ f)(a) = g(a)$, which implies that $h \circ f = g$.

We now prove the equivalence of (ii) and (iii).

(ii) $\to$ (iii) trivial

(iii) $\to$ (ii) consider
$$\begin{array}{ccccc} 0 & \to & C & \xrightarrow{j} & D \\ & & \downarrow g & & \\ & & M & & \end{array}$$
where C is an N-submodule of the right N-module D.

Put $X = \{(C', g') \mid C' \subseteq D$ is a N-submodule containing C and $g': C' \to M$ an N-homomorphism such that $g'|_C = g\}$

As usual we can check that X has a maximal element, say (C_o, g_o). We claim that $C_o = D$. If not, let x be in D such that $x \notin C_o$.

Let $u = \{a \varepsilon N \mid xa \varepsilon C_o\}$. Then u is a right ideal of N. Define $f: u \to M$ by $f(a) = g_o(xa)$ for all a in u. One can check that f is an N-homomorphism. So there exists an m in M such that $f(a) = ma = g_o(xa)$ for all a in u.

Let $f_o': xN + C_o + xN \to M$ be given by

$$f_o'(xr + c_o + xr') = mr + g_o(c_o) + mr'.$$

We wish to show that f_o' is well-defined. To this end we show that f_o' vanishes on zero and that f_o' is additive.

Let $xr + c_o + xr' = 0$ for some r, r' in N, c_o in C_o.

Then $c_o = -xr - xr' = x(-r-r') \varepsilon C_o$ and so $(-r-r')$ is in u.

So $g_o(c_o) = g_o\{x(-r-r')\} = f(-r-r') = m(-r-r') = -mr - mr'$.

Hence $mr + g_o(c_o) + mr' = 0$, thus $f_o'(xr + c_o + xr') = 0$. Secondly let $xr + c_o + xr'$ and $xs + c_o' + xs'$ be any two elements of $xN + C_o + xN$.

Then $f_o'\{(xr+c_o+xr') + (xs+c_o'+xs')\}$

$$= f_o'[(xr+xr')+(-xr'+c_o+xr') + (xs + c_o'-xs) + (xs+xs')]$$

$$= f_o'[x(r+r') + b_o + x(s+s')] \quad \text{where } b_o = (-xr'+c_o+xr') + (xs+c_o'-xs) \varepsilon C_o$$

$$= m(r+r') + g_o[(-xr'+c_o+xr')+(xs+c_o'-xs)] + m(s+s')$$

$$= m(r+r') + g_o(-xr'+c_o+xr') + g_o(xs+c_o'-xs) + m(s+s')$$

$$= f_o'[x(r+r') - xr' + c_o + xr'] + f_o'[xs + c_o' - xs + x(s+s')]$$

$$= f_o'(xr + c_o + xr') + f_o'(xs + c_o' + xs')$$

Hence f_o' is well defined and additive. Now since f_o' is additive and $(N,+)$ is generated by a set of distributive elements, it can be seen that f_o' is N-linear and hence a N-homomorphism.

Moreover $f_o'|_{C_o} = g_o$ and $f_o'|_C = g_o|_C = g$

Now let A be the N-submodule of D generated by $xN + C_o + xN$. Then

$$A = \{\sum_{i=1}^{n} [-d_i + (xr_i + c_{o_i} + xr_i') + d_i] \mid d_i \in D,\ r_i, r_i' \in N,\ c_{o_i} \in C_o, 1 \le i \le n\}$$

Then clearly $C \subseteq C_o \subseteq xN + C_o + xN \subseteq A \subseteq D$. Consider the mapping $f_o'': A \to M$ given by

$$f_o''\left[\sum_{i=1}^{n} -d_i + (xr_i + c_{o_i} + xr_i') + d_i\right] = \sum_{i=1}^{n} f_o'(xr_i + c_{o_i} + xr_i')$$

To show that f_o'' is well defined let us take

$$\sum_{i=1}^{n} [-d_i + (xr_i + c_{o_i} + xr_i') + d_i] = \sum_{i=1}^{m} [-d_i' + (xw_i + c_{o_i}' + xw_i') + d_i'] \text{ in } A$$

$$\text{Then } (xr_1 + c_{o_1} + xr_1') = \sum_{i=1}^{m} [(d_1 - d_i') + (xw_i + c_{o_i}' + xw_i') + (d_i' - d_1)]$$

$$+ [(d_1 - d_n) - (xr_n + c_{o_n} + xr_n') + (d_n - d_1)] + \dots$$

$$\dots + [(d_1 - d_2) - (xr_2 + c_{o_2} + xr_2') + (d_2 - d_1)]$$

But since $-(xr_k + c_{o_k} + xr_k') = (xr_k + c_{o_k} + xr_k')(-1)$ for $2 \le k \le n$

we have that

$$f'_o(xr_1+c_{o_1}+ xr'_1) = \sum_{i=1}^{m} f'_o(xw_i+ c'_{o_i} + xw'_i) - f'_o(xr_n+ c_{o_n}+xr'_n)-\ldots - f'_o(xr_2+c_{o_2}+xr'_2).$$

Hence $\sum_{i=1}^{n} f'_o(xr_i+c_{o_i}+xr'_i) = \sum_{i=1}^{m} f'_o(xw_i+c'_{o_i}+ xw'_i)$, which implies that f''_o is well defined. One can check that f''_o is an N-homomorphism and $f''_o|_C = g$. Hence $(A, f''_o) \varepsilon X$ and $(C_o, g_o) < (A, f''_o)$, which contradicts the choice of (C_o, g_o). Hence $C_o = D$, which proves the result.

§ 2. Construction of an Injective Cover of any right N-module, when N is a left d.g. near-ring.

2.1. We call a near ring module I to be an injective cover of a given near ring module M iff there exists a monomorphism from M into I.

To construct an injective cover for a given near-ring module A we proceed as follows:

Let N be a left d.g. near ring with identity. For any arbitrary set S, let $F_S = \{f:S \to N | f(s) = 0$ for all but finite number of elements in S$\}$.

F_S is an additive group and if for any $f \varepsilon F_S$ and $n \varepsilon N$ we define $f.n$ by $(f.n)(s) = f(s)n$ for all $s \varepsilon S$, then F_S is a right N-module.

For any $s \varepsilon S$ define $f_s: S \to N$ by $f_s(x) = 1$ if $x = s$

$= 0$ if $x \neq s$.

Then $f_s \varepsilon F_S$. Define $\psi : S \to F_S$ by $\psi(s) = f_s$ for all $s \varepsilon S$, then

ψ is one-to-one map and we can say $S \subseteq F_S$ after identifying s with f_s.

We need the following results:

2.2. Lemma : $\sum_{i=1}^{n} (-f_i + \lambda_i s_i + f_i) = 0$ implies that $s_i = 0$ for all $1 \leq i \leq n$, where $f_i \in F_S (1 \leq i \leq n)$, $\lambda_i \in S$, $s_i \in N$ and all λ_i's are distinct.

Proof : We have $0 = \sum_{i=1}^{n} (-f_i + \lambda_i s_i + f_i) = \sum_{i=1}^{n} (-f_i + f_{\lambda_i} s_i + f_i) = \phi(\text{say})$ $\in F_S$. Hence $\phi(\lambda_i) = 0$ for all $1 \leq i \leq n$. So that

$$\{-f_1(\lambda_i) + f_{\lambda_1}(\lambda_i)s_1 + f_1(\lambda_i)\} + \ldots + \{-f_i(\lambda_i) + f_{\lambda_i}(\lambda_i)s_i + f_i(\lambda_i)\} + \ldots$$
$$+ \{-f_n(\lambda_i) + f_{\lambda_n}(\lambda_i)s_n + f_n(\lambda_i)\} = 0,$$

or $\{-f_1(\lambda_i) + f_1(\lambda_i)\} + \ldots + \{-f_i(\lambda_i) + s_i + f_i(\lambda_i)\} + \ldots + \{-f_n(\lambda_i) + f_n(\lambda_i)\} = 0$

or $-f_i(\lambda_i) + s_i + f_i(\lambda_i) = 0$, which implies that $s_i = 0$.

2.3. Corollary : Suppose $\sum_{i=1}^{n} (-f_i + \lambda_i s_i + f_i) = 0$ where $f_i \in F_S$, $\lambda_i \in S$, and $s_i \in N$. Let λ_i be repeated k times, that is $\lambda_i = \lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3} = \ldots = \lambda_{i_k}$, then $\sum_{j=1}^{k} -f_{i_j}(\lambda_i) + s_{i_j} + f_{i_j}(\lambda_i) = 0$ and for non repeated $\lambda_j (j \neq i)$, $s_j = 0$.

Proof : As in the previous Lemma $\phi(\lambda_i) = 0$ implies that $\sum_{j=1}^{k} (-f_{i_j}(\lambda_i) + s_{i_j} + f_{i_j}(\lambda_i)) = 0$ and for non repeated $\lambda_j (j \neq i)$, $\phi(\lambda_j) = 0$ implies that $-f_j(\lambda_j) + s_j + f_j(\lambda_j) = 0$, which means that $s_j = 0$.

Now to construct an injective cover for a right module A over a d.g. near ring N, we first define a module D(A) containing A and satisfying the property:

* If I is a right ideal of N and $f: I \to A$ is a right N-homomorphism then there exists an element g in D(A) such that $f(\lambda) = g\lambda$ for all λ in I.

Let $\Phi = \{(I, f) \mid I$ is a right ideal of N and $f: I \to A$ is a right N-homomorphism$\}$

Then F_Φ is a right N-module. Let G = submodule of $A \times F_\Phi$ generated by the elements $(g(\lambda), -(I,g)\lambda)$ with (I,g) in Φ, λ in I. Then

$$G = \{\sum_{i=1}^{n} [-(a_i,f_i) + (g_i(\lambda_i), -(I_i,g_i)\lambda_i)s_i + (a_i,f_i)] \mid a_i \varepsilon A, f_i \varepsilon F_\Phi, (I_i,g_i) \varepsilon \Phi, \lambda_i \varepsilon I, s_i \varepsilon \text{ generating set of } N\}$$

$$= \{\sum_{i=1}^{n} (-a_i+g_i(\lambda_i)s_i+a_i, -f_i - (I_i,g_i)\lambda_i s_i + f_i)\}$$

Let $D(A) = A \times F_\Phi \mid G$. Define $\phi: A \to D(A)$ by $\phi(a) = (a,0) + G$, then ϕ is an N-homomorphism. We now show that ϕ is one-to-one.

Now $\phi(a) = 0$ implies $(a,0) + G = 0$ i.e. $(a,0) \varepsilon G$.

So that $$(a,0) = \sum_{i=1}^{n} (-a_i + g_i(\lambda_i)s_i + a_i, -f_i - (I_i,g_i)\lambda_i s_i + f_i)$$

$$= (\sum_{i=1}^{n} \{-a_i + g_i(\lambda_i)s_i + a_i\}, \sum_{i=1}^{n} \{-f_i - (I_i,g_i)\lambda_i s_i + f_i\}).$$

Hence $a = \sum_{i=1}^{n} -a_i + g_i(\lambda_i)s_i + a_i$

and $0 = \sum_{i=1}^{n} (-f_i - (I_i, g_i)\lambda_i s_i + f_i)$

If (I_i, g_i) is not repeated for any i, then by 2.2, $\lambda_i s_i = 0$ for i=1,2, ...n. Hence a = 0, which implies that ϕ is one-to-one.

Suppose now that $(g_i(\lambda_i), -(I_i, g_i)\lambda_i)$ is repeated k times, $(g_j(\lambda_j), -(I_j, g_j)\lambda_j)$ is repeated ℓ times and so on, $(g_s(\lambda_s), -(I_s, g_s)\lambda_s)$ is repeated r times. Then

$$(g_i(\lambda_i), -(I_i, g_i)\lambda_i) = (g_{i_1}(\lambda_{i_1}), -(I_{i_1}, g_{i_1})\lambda_{i_1}) = \dots = (g_{i_k}(\lambda_{i_k}), -(I_{i_k}, g_{i_k})\lambda_{i_k})$$

$$\therefore g_i(\lambda_i) = g_{i_1}(\lambda_{i_1}) = g_{i_2}(\lambda_{i_2}) = \dots = g_{i_k}(\lambda_{i_k})$$

and $(I_i, g_i)\lambda_i = (I_{i_1}, g_{i_1})\lambda_{i_1} = (I_{i_2}, g_{i_2})\lambda_{i_2} = \dots = (I_{i_k}, g_{i_k})\lambda_{i_k}$.

Put $(I_i, g_i) = b_i$. Then by 2.3, $0 = \sum_{j=1}^{k} [-f_{i_j}(b_i) - \lambda_i s_{i_j} + f_{i_j}(b_i)]$

We get similar equations for the other repeated (I, g) $\lambda^{'s}$ and we get that $\lambda_k s_k = 0$ for non repeated (I_k, g_k) $\lambda_k^{'s}$. Hence for non repeated $(I_k, g_k)\lambda_k^{'s}$, $g(\lambda_k s_k) = g(\lambda_k)s_k = 0$. So in the expression for a, only the terms corresponding to those indecies, for which $(I_i, g_i)\lambda_i$ are repeated, occur, other terms become zero.

So in the expression for a, some of the $\lambda^{'s}$ will be repeated. We wish to collect together those terms in the expression for a which

have the same λ occuring in them. For example if

$$a = \{-a_1+g_1(\lambda_1 s_1)+a_1\} + \{-a_2+g_2(\lambda_2 s_2)+a_2\} + \{-a_3+g_1(\lambda_1 s_3)+a_3\} + \{-a_4+g_2(\lambda_2 s_4)+a_4\}$$

then we wish to bring the first term and the third term side by side and second term and fourth term side by side. We can do this as follows:

$$\begin{aligned} a &= \{-a_1+g_1(\lambda_1 s_1)+a_1\} + \{-a_2+g_2(\lambda_2 s_2)+a_2\} + \{-a_3+g_1(\lambda_1 s_3)+a_3\} \\ &\quad + \{-a_4+g_2(\lambda_2 s_4)+a_4\} \\ &= \{-a_1+g_1(\lambda_1 s_1)+a_1\} + [\{-a_2+g_2(\lambda_2 s_2)+a_2-a_3\} + g_1(\lambda_1 s_3) \\ &\quad + \{a_3-a_2-g_2(\lambda_2 s_2)+a_2\}] + \{-a_2+g_2(\lambda_2 s_2)+a_2\} \\ &\quad + \{-a_4+g_2(\lambda_2 s_4)+a_4\} \\ &= \{-c_1+g_1(\lambda_1 s_1)+c_1\} + \{-c_2+g_1(\lambda_1 s_3)+c_2\} + \{-c_3+g_2(\lambda_2 s_2)+c_3\} \\ &\quad + \{-c_4+g_2(\lambda_2 s_4)+c_4\} \\ &= \{-c_1+g_1(\lambda_1 t_1)+c_1\} + \{-c_2+g_1(\lambda_1 t_2)+c_2\} + \{-c_3+g_2(\lambda_2 t_1')+c_3\} \\ &\quad + \{-c_4+g_2(\lambda_2 t_2')+c_4\} \end{aligned}$$

where $c_1 = a_1$, $-c_2 = -a_2+g_2(\lambda_2 s_2)+a_2-a_3$, $c_3 = a_2$, $c_4 = a_4$, $t_1 = s_1$, $t_2 = s_3$, $t_1' = s_2$, $t_2' = s_4$ and c_i's are in A for all $i = 1,2,3,4$.

In general $a = \sum_{j=1}^{k} \{-c_{i_j} - g_{i_j}(\lambda_{i_j} t_{i_j}) + c_{i_j}\} + \sum_{p=1}^{\ell} \{-c_{j_p} - g_{j_p}(\lambda_{j_p} t_{j_p}) + c_{j_p}\}$

$$+ \ldots + \sum_{m=1}^{r} \{-c_{s_m} - g_{s_m}(\lambda_{s_m} t_{s_m}) + c_{s_m}\} .$$

Let Q be the subgroup of F_Φ generated by $\{f_{i_1}, f_{i_2}, \ldots, f_{i_k}\}$. Then an arbitrary element of Q is of the form $u = \sum_j \pm \theta_j$ where θ_j is f_{i_r} for some $r(1 \le r \le k)$.

Now let $B_1 = \{\sum_j [u_j(b_i) + x_j t_j - u_j(b_i)] \mid u_j \in Q,\ x_j \in I_i, t_j \in N\}$, and

let $g_1' : B_1 \to A$ be given by

$g_1'[\sum_j \{u_j(b_i) + x_j t_j - u_j(b_i)\}] = \sum_j \{a_j - g_i(x_j t_j) - a_j\}$, where

$a_j = \sum_{r_j \le k} \pm c_{i_{r_j}}$ whenever $u_j = \sum_{r_j \le k} \pm f_{i_{r_j}}$. Then clearly g_1' is additive and $g_1'(x) = -g_i(x)$ for all x in I_i.

To prove that g_1' is well defined, we only have to show that $g_1'(0) = 0$. For this let $y = \sum_{j=1}^{m} [u_j(b_i) + x_j t_j - u_j(b_i)] = 0$.

Then $\sum_{j=1}^{m-1} [u_j(b_i) + x_j t_j - u_j(b_i)] = u_m(b_i) - x_m t_m - u_m(b_i)$

or $-u_m(b_i) + \sum_{j=1}^{m-1} [u_j(b_i) + x_j t_j - u_j(b_i)] + u_m(b_i) = -x_m t_m$.

or $d = \sum_{j=1}^{m-1} [\{(-u_m + u_j)(b_i)\} + x_j t_j + \{(-u_j + u_m)(b_i)\}] = -x_m t_m \in I_i$.

Hence $g_i(-x_m t_m) = g_i(d) = -g_1'(d)$ which implies that

$$g_i(x_m t_m) = g_1'(d) = \sum_{j=1}^{m-1} [\{-a_m+a_j\} - g_i(x_j t_j) + \{-a_j+ a_m\}]$$

$$= [(-a_m+a_1) - g_i(x_1 t_1) +(-a_1+a_m)] + [(-a_m+a_2)- g_i(x_2 t_2) +(-a_2+ a_m)]$$

$$+ \dots + [(-a_m+a_{m-1})- g_i(x_{m-1} t_{m-1}) + (-a_{m-1}+ a_m)]$$

So that $g_i(x_m t_m) = - a_m+ \{a_1- g_i(x_1 t_1)-a_1\} + \{a_2- g_i(x_2 t_2)- a_2\} + \dots +$

$$+ \{a_{m-1} - g_i(x_{m-1} t_{m-1}) - a_{m-1}\} + a_m,$$

which implies that $g_1'(y) = \sum_{j=1}^{m} \{a_j - g_i(x_j t_j) - a_j\} = 0$.

Since $\sum_{j=1}^{k} \{-f_{i_j}(b_i)- \lambda_i s_{i_j} + f_{i_j}(b_i)\} = 0$ is in B_1, we have

$$0 = g_1' [\sum_{j=1}^{k}\{-f_{i_j}(b_i) -\lambda_i s_{i_j} + f_{i_j}(b_i)\}] = \sum_{j=1}^{k} \{-c_{i_j} - g_i(\lambda_i s_{i_j}) + c_{i_j}\}$$

$$= \sum_{j=1}^{k} \{-c_{i_j} - g_{i_j}(\lambda_{i_j})s_{i_j} + c_{i_j}\} \quad \text{since } g_i(\lambda_i) = g_{i_j}(\lambda_{i_j}) \text{ for all}$$

$j = 1,2,\dots k$. Similarly we can show that the other summations in the expression for a are zero. Hence $a = 0$, giving that ϕ is one-to-one. Hence A is contained in D(A). We now show that D(A) has the property *. For this let I be a right ideal of N and f: I→ A a right N-homomorphism, then for all λ in $I, f(\lambda) = (f(\lambda), 0) + G = (0, (I, f)\lambda)+ G = g\lambda$.

Let Ω be the least infinite ordinal number whose cardinal is larger than that of the near ring N. We define $Q_\alpha(A)$ for $\alpha \leq \Omega$ by

transfinite induction as follows:

$Q_1(A) = D(A)$; if $\alpha = \beta+1$ then $Q_\alpha(A) = D(Q_\beta(A))$; if α is a limiting ordinal then $Q_\alpha(A)$ is the union of Q_β with $\beta < \alpha$. We now prove that Q_Ω (A) is injective. Let $f:I \to Q_\Omega(A)$ where I is a right ideal of N. Then because of the choice of Ω we have $f(I) \subseteq Q_\alpha(A)$ for some $\alpha < \Omega$. Then by * there is an element g in $D(Q_\alpha(A)) = Q_{\alpha+1}(A) \subseteq Q_\Omega(A)$ with $f(\lambda) = g\lambda$ for all λ in I. Thus $Q_\Omega(A)$ is injective and $A \subseteq Q_\Omega(A)$. Thus $Q_\Omega(A)$ is an injective cover of A.

§3. Injective Hull

Throughout this section N will denote an arbitrary near-ring, not necessarily d.g.

Although the results in this section and their proofs are similar to those in the ring theory case, they are included here for self sufficiency.

3.1. We call a monomorphism $\kappa : M \to B$ direct if there exists a homomorphism $\pi : B \to M$ such that $\pi \circ \kappa = 1$.

Remark 1: $\kappa : M \to B$ direct implies that $B = \ker\pi + \operatorname{Im}\kappa$ and

$$\ker\pi \cap \operatorname{Im}\kappa = 0.$$

Remark 2: Let I be an injective module such that $I = K + L$ for some sub-module K of I and some N-subgroup L of I where $K \cap L = 0$, then K and L both are injective.

3.2. Proposition : M is injective iff every monomorphism $\kappa : M \to B$ is direct.

Proof : Suppose every monomorphism $M \to B$ is direct. In particular mono $\kappa : M \to I$ is direct, where I is an injective cover of M. This implies that $I = K + \kappa(M)$ where K is a submodule of I such that $K \cap \kappa(M) = 0$. Hence $\kappa(M) \cong M$ is injective.

Conversely let M be an injective module. Let $\kappa : M \to B$ be any monomorphism. Then by injectivity of M there exists a homomorphism $\pi : B \to M$ with $\pi \circ \kappa = 1$.

3.3. Definition : A module P containing M (not necessarily as a submodule) is called a weakly essential extension of M provided every nonzero submodule of P has nonzero intersection with M.

3.4. Lemma : Let P be a weakly essential extension of M and let I be an injective module containing M, then the identity mapping of M can be extended to a monomorphism of P into I.

Proof : Since I is injective, the identity mapping of M can be extended to a N-homomorphism $\phi : P \to I$, hence $\phi^{-1}(0) \cap M = 0$. Since P is weakly essential extension of M, $\phi^{-1}(0) = 0$.

3.5. Proposition : M is injective if and only if M has no proper weakly essential extension.

Proof : Suppose M is injective and P is weakly essential extension of M. Then $P = K + M$ for some submodule K of P such that $K \cap M = 0$. This implies $K = 0$, hence $P = M$. Therefore P is not a proper extension.

Conversely assume that M has no proper weakly essential extension

Let I be an injective module containing M, and let M' be a submodule of I which is maximal with respect to the property that $M \cap M' = 0$. We shall see that I/M' is weakly essential extension of $(M'+M)/M' \cong M$. By assumption, $I/M' \cong M$, hence $I = M'+M$. Since I is injective, we have M is injective.

It remains to show that I/M' is an essential extension of $(M'+M)/M'$. Let K/M' be a submodule of I/M' ($\dashrightarrow$ K is a submodule of I) such that $K/M' \cap (M'+M)/M' = (0)$, that is $K \cap (M'+M) \subseteq M'$. Then $K \cap M \subseteq M \cap M' = 0$, hence by the maximality of M', $K \subseteq M'$. Thus $K/M' = 0$ and our proof is complete.

To say that P is a maximal weakly essential extension of M means that P is weakly essential extension of M and that no proper extension of P is a weakly essential extension of M.

3.6. Proposition : Every module M has a maximal weakly essential extension P. This is unique in the following sense: If S is another maximal weakly essential extension of M, then the identity mapping of M can be extended to an isomorphism of S onto P.

Proof : Let I be an injective module containing M. The union of any simply ordered family of weakly essential extensions of M in I is also weakly essential extension. Hence by Zorn's lemma, M has a maximal weakly essential extension P in I.

Now let S be any weakly essential extension of M containing P, not necessarily in I. S is also a weakly essential extension of P

($U \neq 0$ submodule of $S \Longrightarrow 0 \neq M \cap U \subseteq P \cap U$). By Lemma II.3.4 the identity mapping of P can be extended to a monomorphism $\phi : S \to I$. Now $\phi(S) \cong S$ is a weakly essential extension of P and is in I. Hence $\phi(S) \subseteq P \Rightarrow \phi(S) \cong S \subseteq P \subseteq S$. So $P = S$. Therefore P is a maximal weakly essential extension of M, not just in I, but absolutely.

Any weakly essential extension T of P is a weakly essential extension of M. For, $U \neq 0$ submodule of $T \Longrightarrow 0 \neq U \cap P$ submodule of P. So $0 \neq M \cap (U \cap P) = U \cap (M \cap P) = U \cap M$. Therefore P has no proper weakly essential extension, hence is injective, by Proposition II.3.5. If S is any weakly essential extension of M, by II.3.4 we can extend the identity mapping of M to a monomorphism of S into P. If S is maximal, this is an isomorphism.

3.7. Proposition : Let P be an extension of M. The following statements are equivalent :

(1) P is a maximal weakly essential extension of M.

(2) P is a weakly essential extension of M and is injective.

(3) P is a minimal injective extension of M.

P is called the injective hull of M.

Proof : Assume (1). Then P has no proper weakly essential extension, hence is injective. Thus (1) implies (2).

Assume (2) and suppose $M \subset I \subset P$, I injective. Then $P = K + I$ for some submodule K of P with $K \cap I = 0$. But P is a weakly essential extension

of I, hence $K = 0$, so that $P = I$. Thus (2) implies (3).

Assume (3) and let S be a maximal weakly essential extension of M in P. Then S is injective (as above), hence $S = P$. Thus (3) implies (1).

CHAPTER III

CLASSICAL NEAR RING AND MODULE OF QUOTIENTS

Let N be a near ring containing some non zero-divisors and S be a a semi group of some distributive non zero-divisors of N. An extension near ring C of N is called a classical near ring of left (right) quotients of N with respect to S iff

(i) $1 \in C$.

(ii) elements of S are invertible in C.

(iii) for all x in C there exists λ in S such that λx ($x\lambda$) belongs to N.

Here we construct (using Asano's method [1]) classical near rings of left and right quotients of a left unitary near ring N with respect to S(a semi-group of some distributive nonzero divisors of N, containing 1), and show that classical near ring of left quotients, of N with respect to S, is unique upto isomorphism.

In general we do not know whether all the classical near rings of right quotients of N with respect to S are isomorphic. However we have shown that a classical near ring Q_1 of right quotients of N with respect to S, is unique if it has the property that λ^{-1} is distributive in Q_1 whenever λ is in S.

It is also shown that the concepts of classical near rings of left and right quotients are quite independent of each other.

Analogous to the construction of the classical near ring Q' of right quotients, we give the construction of a module $\overline{M}$ of quotients for any right module M over a d.g. near-ring N.

We show that : Whenever M is S-free ($ms = 0$ implies $m=0$ for every $m \in M$ and every $s \in S$), $\overline{M}$ is an injective Q'- module (or injective N-module) iff every N-homomorphism $f: I \to M$, I- a right ideal of N, can be extended to an N-homomorphism $g: J \to M$, where J is a right ideal of N containing I and containing an element of S.

As a Corollary to the above result we get: Q'_N is injective iff for every N-homomorphism $f: I \to N$, I-a right ideal of N, there exists an N-homomorphism $g: J \to N$, where J is a right ideal of N containing I and containing an element of S and g is such that $g(x) = f(x)$ for all x in I.

§ 1. Classical near ring of right quotients.

1.1. A near ring N is said to have common right multiple property (C.R.M.P.) iff for any a in N, λ in S there exists a' in N and λ' in S such that $a \lambda' = \lambda a'$.

We need the following :

1.2. Lemma : If N satisfies C.R.M.P. with respect to S then for finitely many elements, $\lambda_1, \lambda_2, \ldots, \lambda_n$ of S there exists μ in S such that $\mu = \lambda_1 c_1 = \lambda_2 c_2 = \ldots = \lambda_n c_n$ $(c_i \in N)$.

Proof : The proof is by induction on n. Suppose the result is true for n-1 elements, that is there exists μ' in S and c_i' $(i=1,2,\ldots,n-1)$ in N such that $\mu' = \lambda_1 c_1' = \lambda_2 c_2' = \ldots = \lambda_{n-1} c_{n-1}'$. Given μ' in S and λ_n there exist ν in S and c' in N with $\mu = \mu'\nu = \lambda_n c'$, that is $\mu = \lambda_1 c_1'\nu = \lambda_2 c_2'\nu = \ldots = \lambda_{n-1} c_{n-1}'\nu = \lambda_n c'$, or $\mu = \lambda_1 c_1 = \lambda_2 c_2 = \ldots = \lambda_n c_n$.

1.3. Definition : A right N-subgroup is called regular if it contains atleast one element of S.

1.4. We note that if N has C.R.M.P. with respect to S, then

(1) The intersection of finite number of regular right N-subgroups is again a regular right N-subgroup.

(2) For any regular right N-subgroup A and for any a in N there exists μ in S such that $a\mu$ belongs to A .

One can easily see that if N has a classical near ring of right quotients with respect to S then N has C.R. M.P. with respect to S. We do not know whether the converse holds. However we have the following:

1.5. Theorem : Let N be a left near ring satisfying the following conditions:

(i) C.R.M.P. : Given a in N, λ in S there exists a' in N and λ' in S such that $a\lambda' = \lambda a'$.

(ii) Whenever a and λ both belong to S, a' is distributive.

Then N has a classical near ring of right quotients.

Proof : For each $\lambda \in S$, λN is a regular right N-subgroup. Set $Y_\lambda = \{\theta \mid \theta : \lambda N \to N$ such that $\theta(\lambda r\mu) = \theta(\lambda r)\mu \;\forall\, r \in N$ and all distributive element μ in $N\}$. Let $Y = \bigcup_{\lambda \in S} Y_\lambda$. In Y define the following relation : $\theta_1 \sim \theta_2$ iff there exists λ in $\lambda_1 N \cap \lambda_2 N \cap S$ such that $\theta_1(\lambda) = \theta_2(\lambda)$ where $\theta_i \in Y_{\lambda_i}$ (i=1,2). Clearly '$\sim$' is a reflexive and symmetric relation. To show that it is transitive also, suppose $\theta_1 \sim \theta_2$ and $\theta_2 \sim \theta_3$ where $\theta_i \in Y_{\lambda_i}$ (i=1,2,3). This implies there exists $\mu_1 \in \lambda_1 N \cap \lambda_2 N \cap S$ and $\mu_2 \in \lambda_2 N \cap \lambda_3 N \cap S$ such that $\theta_1(\mu_1) = \theta_2(\mu_1)$ and $\theta_2(\mu_2) = \theta_3(\mu_2)$. In view of C.R.M.P. we have that there exists $x' \in N$ and $\lambda' \in S$ such that $\mu_1 \lambda' = \mu_2 x'$. Since μ_1, μ_2 both are in S, x' is distributive. Since θ_1 and θ_3 coincide on $\mu = \mu_2 x' = \mu_1 \lambda' \in \mu_1 N \cap \mu_2 N \cap S \subseteq \lambda_1 N \cap \lambda_2 N \cap \lambda_3 N \cap S$, we have $\theta_1 \sim \theta_3$.

Let Q' denote the set of equivalence classes of Y with respect to the equivalence relation defined above. Addition in Q' is defined as follows: Let $[\theta_1], [\theta_2]$ be any two elements of Q' where θ_i has the domain $\lambda_i N$ (i=1,2). Then there exists $x' \in N$ and $\lambda' \in S$ such that $\nu = \lambda_1 \lambda' = \lambda_2 x'$. Define $\phi : \nu N \to N$ by $\phi(\nu r) = \theta_1(\nu r) + \theta_2(\nu r)$. Then $\phi \in Y_\nu$. Now if $x'' \in N$, $\lambda'' \in S$ are another set of elements such that $\nu' = \lambda_1 \lambda'' = \lambda_2 x'' \in \lambda_1 N \cap \lambda_2 N \cap S$, then for $\psi : \nu' N \to N$ given by $\psi(\nu' r) = \theta_1(\nu' r) + \theta_2(\nu' r)$ we have that $\psi \in Y_{\nu'}$, and that $[\phi] = [\psi]$. Hence without ambiguity we may write $\theta_1 + \theta_2$ for ϕ. Thus $\theta_1 + \theta_2 : \nu N \to N$ is given by $(\theta_1 + \theta_2)(\nu r) = \theta_1(\nu r) + \theta_2(\nu r)$. We write $[\theta_1] + [\theta_2] = [\theta_1 + \theta_2]$. To show that the addition is

well defined take $[\theta_1']$, $[\theta_2']$ in Q' such that $\theta_1 \sim \theta_1'$ and $\theta_2 \sim \theta_2'$ where $\theta_i' \in Y_{\lambda_i'}$ $(i=1,2)$. Then $[\theta_1'] + [\theta_2'] = [\theta_1' + \theta_2']$ where $\theta_1' + \theta_2'$: $\nu'N \to N$. Here $\nu' = \lambda_1'\lambda'' = \lambda_2'x'' \in \lambda_1'N \cap \lambda_2'N$ for some $\lambda'' \in S$ and some $x'' \in N$.

Now $\theta_i \sim \theta_i'$ implies there exists $\mu_i \in \lambda_i N \cap \lambda_i'N \cap S$ such that $\theta_i(\mu_i) = \theta_i'(\mu_i)$ $(i = 1,2)$. In view of C.R.M.P. we have that there exists $b \in N$ and $s \in S$ such that $\mu_1 s = \mu_2 b = \mu \in S$ and there exists $b' \in N$ and $s' \in S$ such that $\nu s' = \nu' b' = t_1 \in S$. Given μ , t_1 we have the existence of $a \in N$ and $t \in S$ with $\mu t = t_1 a = \nu'' \in \mu N \cap t_1 N \cap S \subseteq \nu N \cap \nu' N \cap S$. Since $(\theta_1+\theta_2)(\nu'') = (\theta_1'+\theta_2')(\nu'')$ we have that $[\theta_1+\theta_2] = [\theta_1' + \theta_2']$, which implies that the addition is well defined.

It is easy to see that $(Q', +)$ is a group.

To define multiplication in Q', take $[\theta_1],[\theta_2]$ any two elements of Q' with $\theta_i \in Y_{\lambda_i}$ $(i=1,2)$. Define $\theta_1' : \lambda_1 N \to N$ by $\theta_1'(\lambda_1 r) = \theta_1(\lambda_1)r$, then $\theta_1' \in Y_{\lambda_1}$ and $[\theta_1'] = [\theta_1]$. Now let $\mu \in \lambda_2 N \cap S$ such that $\theta_2(\mu) \in \lambda_1 N$. Such a μ exists since for any $\tau \in \lambda_2 N \cap S$ and $k \in \lambda_1 N \cap S$ we have that $\theta_2(\tau)\lambda' = k x' \in \lambda_1 N$ for some $\lambda' \in S$ and some $x' \in N$. Since λ' is distributive we have $\theta_2(\tau\lambda') \in \lambda_1 N$. Clearly $\tau\lambda' \in \lambda_2 N \cap S$, take $\mu = \tau\lambda'$.

Let $\phi : \mu N \to N$ be given by $\phi(\mu r) = \theta_1'(\theta_2(\mu)r)$. Then $\phi \in Y_\mu$ and if $\mu' \in \lambda_2 N \cap S$ is another element such that $\theta_2(\mu') \in \lambda_1 N$ and if $\psi : \mu'N \to N$ is given by $\psi(\mu'r) = \theta_1'(\theta_2(\mu')r)$ then $\phi = \psi$

on $S \cap \mu N \cap \mu' N$ and hence $[\phi] = [\psi]$. So without ambiguity we can write ϕ as $\theta_1' \circ \theta_2$. We write $[\theta_1]\,[\theta_2] = [\theta_1' \circ \theta_2]$. To show that multiplication is well defined, take $[\phi_1]$, $[\phi_2]$ in Q' such that $\phi_i \in Y_{\mu_i}$ $(i=1,2)$ and $\theta_i \sim \phi_i$. As before define $\phi_1' : \mu_1 N \to N$ by $\phi_1'(\mu_1 r) = \phi_1(\mu_1)\,r$ and let $\mu' \in \mu_2 N \cap S$ such that $\phi_2(\mu') \in \mu_1 N$. Let ψ : $\mu' N \to N$ be given by $\psi(\mu' r) = \phi_1'(\phi_2(\mu')r)$. Then $[\phi_1]\,[\phi_2] = [\psi] = [\phi_1' \circ \phi_2]$.

Now $\theta_i \sim \phi_i$ implies there exists $s_i \in \lambda_i N \cap \mu_i N \cap S$ such that $\theta_i(s_i) = \phi_i(s_i)$ ($i=1,2$). By C.R.M.P. there exist elements a, x', x'' in N and elements t, λ', λ'' in S such that $s_1 t = s_2 a = b \in S$, $\mu\lambda' = \mu' x' = \nu \in S$ and $bx'' = \nu\lambda'' = \nu' \in S$. Moreover a, x', x'' are distributive and $\theta_2(\nu') = \phi_2(\nu')$.

Next let p be an element in $\nu' N \cap S$ such that $\theta_2(p) \in s_1 N$. Now $p \in \nu' N$ implies $p = \nu' r$ for some $r \in N$. Since p and ν' are in S, r is distributive. Hence $\phi_2(p) = \theta_2(p) = \theta_2(\nu' r) = s_1 n_1$ for some $n_1 \in N$. Also $\phi(p) = \phi(\nu' r) = \phi(\mu\lambda'\lambda'' r) = \theta_1'(\theta_2(\mu)\lambda'\lambda'' r) = \theta_1'(\theta_2(\nu')r) = \theta_1'(\theta_2(p)) = \theta_1'(s_1 n_1) = \theta_1'(\lambda_1 r_1 n_1) = \theta_1(\lambda_1) r_1 n_1 = \theta_1(\lambda_1 r_1) n_1 = \theta_1(s_1) n_1$. Since $s_i = \lambda_i r_i$ for some $r_i \in N (i=1,2)$ and since s_i, λ_i both are in S, r_i is distributive in N $(1=1,2)$.

Similarly $\psi(p) = \theta_1(s_1) n_1$. Since $p \in \mu N \cap \mu' N \cap S$ we have $[\phi] = [\psi]$ that is the multiplication is well defined.

It is a routine matter to check that Q' is a left near ring.

Now $\Phi : N \to Q'$ given by $\Phi(c) = [\phi_c]$ is a near ring isomorphism where $\phi_c : N \to N$ is given by $\phi_c(x) = cx$. After identifying N with $\Phi(N)$ we have that $N \subsetneq Q'$. Moreover since $1 \in N$, $[\phi_1] \in Q'$ and it is the identity element of Q'.

We note that for any $\lambda \in S$, $\theta : \lambda N \to N$ given by $\theta(\lambda x) = x$ belongs to Y_λ and $[\phi_\lambda][\theta] = [\theta][\phi_\lambda] = [\phi_1]$, showing that the elements of S are invertible in Q'.

Lastly, take $q \in Q'$. Then $q = [\psi]$ for some $\psi \in Y_\lambda$ and some $\lambda \in S$. Recall that $\psi' : \lambda N \to N$ is given by $\psi'(\lambda n) = \psi(\lambda)n$ for all $n \in N$. It is easy to see that $[\psi][\phi_\lambda] = [\psi' \circ \phi_\lambda] = [\phi_{\psi(\lambda)}] \in \Phi(N)$. Hence for any $q \in Q'$ there exists λ in S such that $q\lambda \in N$. Thus we have shown that Q' is a classical near ring of right quotients of N with respect to S.

A natural question arises: Is Q' a d.g. near ring whenever N is a d.g. near ring ? We have shown here that whenever N satisfies

(ii)' for any $\lambda \in S$, $r \in N$, λr distributive in N implies that r is distributive in N.

Then Q' is d.g. whenever N is d.g.

<u>Remark</u> : Condition (ii)' implies condition (ii) of Theorem III.1.5 and so whenever conditions (i) and (ii)' are satisfied in a left near ring N, Q' exists.

For a left near ring N satisfying (i) and (ii)' we prove the following results.

1.6. Proposition : If λ belongs to S then λ is distributive in Q'.

Proof : Let $[\theta_1]$, $[\theta_2]$ be arbitrary elements of Q' where $\theta_i \in Y_{\lambda_i}$ $(i = 1,2)$. For each $\lambda \in S$, we wish to show that $\{[\theta_1] + [\theta_2]\} [\phi_\lambda] = [\theta_1][\phi_\lambda] + [\theta_2] [\phi_\lambda]$, that is $[(\theta_1+\theta_2)' \circ \phi_\lambda]=[\theta_1' \circ \phi_\lambda + \theta_2' \circ \phi_\lambda]$.

In view of C.R.M.P. there exist elements $x' \in N$ and $\lambda' \in S$ such that $\lambda_1 x' = \lambda_2 \lambda' = \nu \in S$, x' is distributive since λ_1 and λ_2 both belong to S. The domain of $\theta_1+ \theta_2$ is νN. Let $\mu \in S$ such that $\phi_\lambda(\mu) \in \nu N$. Now $\phi_\lambda(\mu) \in \nu N \Longrightarrow \lambda\mu = \nu r$ for some r in N, since λ, ν are in S, r is distributive.

Now let $\mu_i \in S$ such that $\phi_\lambda(\mu_i) \in \lambda_i N$ $(i = 1,2)$. Say $\lambda\mu_i = \phi_\lambda(\mu_i) = \lambda_i r_i$ for some distributive r_i, since λ and λ_i both are in S. Given μ_1, μ_2 in S there exist elements $x'' \in N$, $\lambda'' \in S$ with x'' distributive, such that $\mu_1 x'' = \mu_2 \lambda'' = \nu' \in S$. Again there exist elements $a \in N$, $s \in S$ with a distributive, such that $b = \mu a = \nu' s \in S$. It is easy to see that $(\theta_1+ \theta_2)' \circ \phi_\lambda(b) = \theta_1' \circ \phi_\lambda (b) + \theta_2' \circ \phi_\lambda(b)$, from which the result follows.

1.7. Corollary : For each λ in S, λ^{-1} is distributive in Q'.

Proof : For any q_1, q_2 in Q', $(q_1+q_2)\lambda^{-1}\lambda = q_1+q_2 = q_1\lambda^{-1}\lambda + q_2\lambda^{-1}\lambda$ $= (q_1\lambda^{-1} + q_2 \lambda^{-1})\lambda$. Hence $(q_1+ q_2)\lambda^{-1} = q_1\lambda^{-1} + q_2\lambda^{-1}$.

1.8. Corollary : Any distributive element of N is distributive in Q' also.

Proof : Suppose $q_1 = [\theta_1]$, $q_2 = [\theta_2]$ be any two elements of Q' and x be any distributive element of N. If $\lambda_i N$ is the domain of $\theta_i (i = 1,2)$, we have the existence of $x' \varepsilon N$, $\lambda' \varepsilon S$ such that $\lambda_1 x' = \lambda_2 \lambda' = \nu \varepsilon S$, x' being distributive. Let $\mu \varepsilon S$ such that $\phi_x(\mu) = x\mu \ \varepsilon \ \nu N$ that is $x\mu = \nu r$ for some distributive r in N. Then μN is the domain of $(\theta_1 + \theta_2)' \circ \phi_x$.

Again let $\mu_i \varepsilon S$ such that $\phi_x(\mu_i) = x\mu_i = \lambda_i r_i \varepsilon \lambda_i N$ $(i=1,2)$ for some distributive $r_i \varepsilon N$. We have the existence of $x'' \varepsilon N$ and $\lambda'' \varepsilon S$ such that $\mu_1 x'' = \mu_2 \lambda'' = \nu' \varepsilon S$, x'' being distributive. Then $\nu' N$ is the domain of $\theta_1' \circ \phi_x + \theta_2' \circ \phi_x$. We also have the existence of $a \varepsilon N$ and $s \varepsilon S$, with a distributive, such that $\mu a = \nu' s = \nu'' \varepsilon S$. Since $(\theta_1 + \theta_2)' \circ \phi_x(\nu'') = \theta_1' \circ \phi_x(\nu'') + \theta_2' \circ \phi_x(\nu'')$, we have the required result.

1.9. Proposition : Whenever $(N, +)$ is generated by a set of distributive elements, so is $(Q', +)$.

Proof : $q \varepsilon Q' \Longrightarrow q = a\lambda^{-1}$ for some $a \varepsilon N$ and some $\lambda \varepsilon S$.

$$= (s_1 + s_2 + \ldots + s_n)\lambda^{-1}$$

$$= s_1 \lambda^{-1} + s_2 \lambda^{-1} + \ldots + s_n \lambda^{-1} \quad \text{by Cor.III.1.7}$$

where $s_i (1 \leq i \leq n)$ belongs to the generating set of $(N, +)$.

Since in view of Cor.III.1.8 each $s_i \lambda^{-1} (1 \leq i \leq n)$ is distributive we have the required result.

One would like to know if all classical left near rings of right quotients of N with respect to S are isomorphic. We do not know the answer in full generality. However, one can prove

1.10. Theorem : If Q_1 is a classical left near ring of right quotients of N such that whenever λ is in S, λ^{-1} is distributive in Q_1, then Q_1 is isomorphic to Q'.

§ 2. Classical near ring of left quotients: In this section we will give a necessary and sufficient condition for the existence of a classical left near ring of left quotients of N and prove that a classical left near ring of left quotients of N is unique upto isomorphism.

2.1. Definition : A near ring N is said to have common left multiple property (C.L.M.P.) with respect to S if for any $a \in N$ and $\lambda \in S$ there exists $a' \in N$ and $\lambda' \in S$ such that $a'\lambda = \lambda' a$.

2.2. Theorem : A left near ring N has a classical left near ring of left quotients with respect to S iff it satisfies C.L.M.P. with respect to S.

Proof : Necessasity follows immediately from definition. To prove sufficiency assume that N satisfies C.L. M.P. with respect to S. For each $\lambda \in S$, $N\lambda$ is a regular left N-subgroup. Set $X_\lambda = \{\theta | \theta : N\lambda \to N$ such that $(rx)\theta = r(x)\theta$ for all $r \in N$ and all $x \in N\lambda\}$. Put $X = \bigcup_{\lambda \in S} X_\lambda$.

We say $\theta_1 \sim \theta_2$ iff there exists $\lambda \in N\lambda_1 \cap N\lambda_2 \cap S$ such that $(\lambda)\theta_1 = (\lambda)\theta_2$ where $N\lambda_i$ is the domain of θ_i(i=1,2). As before $\sim$

is an equivalence relation. Let C' denote the set of equivalence classes of X with respect to the above defined equivalence relation.

Addition and multiplication in C' are defined as follows: Let $[\theta_1], [\theta_2]$ be elements of C' where $N\lambda_i$ is the domain of θ_i $(i=1,2)$. In view of C.L.M.P. there exists $x' \in N$ and $\lambda' \in S$, such that $\nu = \lambda'\lambda_1 = x'\lambda_2 \in N\lambda_1 \cap N\lambda_2 \cap S$. Define $\phi : N\nu \to N$ by $(r\nu)\phi = (r\nu)\theta_1 + (r\nu)\theta_2$ for all $r \in N$. As in the proof of Theorem III.1.5 it can be easily seen that the addition is well defined and that $(C', +)$ is a group.

Let $[\theta_1]$, $[\theta_2]$ be in C' with $\theta_i \in X_{\lambda_i}$ $(i = 1,2)$. Suppose $\mu \in N\lambda_1 \cap S$ such that $(\mu)\theta_1 \in N\lambda_2$ and define $\phi : N\mu \to N$ by $(r\mu)\phi = ((r\mu)\theta_1)\theta_2$. Then $\phi \in X_\mu$ and we write $[\theta_1]\ [\theta_2] = [\phi]$. As in the proof of III. 1.5 it can be easily seen that C' is a classical left near ring of left quotients of N with respect to S.

Now let B be another classical left near ring of left quotients of N with respect to S. For any $b \in B$ there exists $\lambda \in S$ and $x \in N$ such that $b = \lambda^{-1}x$. Let $\psi : B \to C'$ be given by $(b)\psi = (\lambda^{-1}x)\psi = [\theta]$ where $\theta : N\lambda \to N$ is given by $(y\lambda)\theta = yx$ for all $y \in N$. Let $\lambda_1^{-1}x_1 = \lambda_2^{-1}x_2 \in B$ and $\theta_i : N\lambda_i \to N$ be given by $(y\lambda_i)\theta_i = yx_i$ $(i = 1,2)$ for all $y \in N$. In view of C.L.M.P. there exists $x' \in N$ and $\lambda' \in S$ such that $x'\lambda_1 = \lambda'\lambda_2 = \mu \in N\lambda_1 \cap N\lambda_2 \cap S$. Now $\lambda_1^{-1}x_1 = \lambda_2^{-1}x_2$ implies that $(x'\lambda_1)(\lambda_1^{-1}x_1) = (\lambda'\lambda_2)(\lambda_2^{-1}x_2)$ that is $x'x_1 = \lambda'x_2$. Since $(\mu)\theta_1 = x'x_1$ and $(\mu)\theta_2 = \lambda'x_2$ we have $[\theta_1] = [\theta_2]$, which

implies that ψ is well defined.

Take $\lambda_1^{-1}x_1$ and $\lambda_2^{-1}x_2$ in B. In view of C.L.M.P. there exists $x' \varepsilon N$ and $\lambda' \varepsilon S$ with $x'\lambda_1 = \lambda'\lambda_2 = \nu \varepsilon S$. Then, $(\lambda_1^{-1}x_1 + \lambda_2^{-1}x_2)\psi$ $= (\nu^{-1}\nu(\lambda_1^{-1}x_1 + \lambda_2^{-1}x_2))\psi = (\nu^{-1}(\nu\lambda_1^{-1}x_1 + \nu\lambda_2^{-1}x_2))\psi$. By the choice of ν we have that $\nu\lambda_i^{-1}x_i \varepsilon N$ $(i = 1,2)$ and so $\nu\lambda_1^{-1}x_1 + \nu\lambda_2^{-1}x_2 \varepsilon N$. Consider $\phi : N\nu \to N$ given by $(x\nu)\phi = x(\nu\lambda_1^{-1}x_1 + \nu\lambda_2^{-1}x_2)$. Then clearly $(\lambda_1^{-1}x_1 + \lambda_2^{-1}x_2)\psi = [\phi]$. Moreover $(\lambda_i^{-1}x_i)\psi = [\theta_i]$ where $\theta_i : N\lambda_i \to N$ is given by $(y\lambda_i)\theta_i = yx_i$ $(i=1,2)$. Since $\nu = x'\lambda_1 = \lambda'\lambda_2$ we have that $\nu \varepsilon N\lambda_1 \cap N\lambda_2$. Hence for any $\mu \varepsilon S$, $(\mu\nu)(\theta_1 + \theta_2) = (\mu\nu)\theta_1 + (\mu\nu)\theta_2 = (\mu x'\lambda_1)\theta_1 + (\mu\lambda'\lambda_2)\theta_2 =$

$\mu x' x_1 + \mu\lambda' x_2 = \mu(x'x_1 + \lambda' x_2)$. But $(\mu\nu)\phi = \mu(\nu\lambda_1^{-1}x_1 + \nu\lambda_2^{-1}x_2) =$ $\mu(x'x_1 + \lambda'x_2)$; thus $(\mu\nu)\phi = (\mu\nu)(\theta_1 + \theta_2)$ for $\mu\nu \varepsilon N\nu \cap S \subseteq$ $N\lambda_1 \cap N\lambda_2 \cap S$. Hence $[\phi] = [\theta_1 + \theta_2] = (\lambda_1^{-1}x_1)\psi + (\lambda_2^{-1}x_2)\psi$, which proves that ψ is additive. To show that $(\lambda_1^{-1}x_1\lambda_2^{-1}x_2)\psi = (\lambda_1^{-1}x_1)\psi(\lambda_2^{-1}x_2)\psi$ we first note that $x_1\lambda_2^{-1}$ belongs to C' and so there exists s in S and b in N such that $x_1\lambda_2^{-1} = s^{-1}b$. So $(\lambda_1^{-1}x_1\lambda_2^{-1}x_2)\psi = (\lambda_1^{-1}s^{-1}bx_2)\psi$ $= \{(s\lambda_1)^{-1}(bx_2)\}\psi = [\theta]$ where $\theta : N(s\lambda_1) \to N$ is given by $(ys\lambda_1)\theta$ $= ybx_2$ for all y in N. Thus to show that $(\lambda_1^{-1}x_1\lambda_2^{-1}x_2)\psi = (\lambda_1^{-1}x_1)\psi(\lambda_2^{-1}x_2)\psi$ we must show that $[\theta] = [\theta_1][\theta_2] = [\theta_1 \cdot \theta_2]$ where $\theta_1 \cdot \theta_2 : N\mu \to N$, $\mu = c\lambda_1$ being in $N\lambda_1 \cap S$ such that $(\mu)\theta_1$ belongs to $N\lambda_2$. Since N satisfies C.L.M.P. with respect to S, we have that there exist x' in N, λ' in S such that $x's\lambda_1 = \lambda'\mu = \gamma \varepsilon N\mu \cap N(s\lambda_1) \cap S$.

Moreover $(\lambda'\mu)\theta_1 \cdot \theta_2 = (\lambda'(\mu)\theta_1)\theta_2 = (\lambda'(c\lambda_1)\theta_1)\theta_2 = (\lambda'c\,x_1)\theta_2 = \lambda' y\,x_2$,

since $(\mu)\theta_1 = y\lambda_2$ is in $N\lambda_2$ for some y in N. Since $x's\lambda_1 = \lambda'\mu = \lambda$ we have that $x's = \lambda'c$ and so $x's\,x_1 = \lambda'c\,x_1$. But by the choice of and b we have $sx_1 = b\lambda_2$ or $x's\,x_1 = x'b\,\lambda_2 = \lambda'\,c\,x_1$. Thus $(\lambda'c\,x_1)\theta_2 = (x'b\,\lambda_2)\theta_2 = x'b\,x_2$; that is $(x's\,\lambda_1)(\theta_1 \circ \theta_2) = (\lambda'\mu)(\theta_1 \circ \theta_2) = x'bx_2 = (x's\,\lambda_1)\theta$. Since $x's\,\lambda_1$ is in S, we have that $[\theta] = [\theta_1 \circ \theta_2]$. Hence $(\lambda_1^{-1}x_1 \circ \lambda_2^{-1}x_2)\psi = (\lambda_1^{-1}x_1)\psi\,(\lambda_2^{-1}x_2)\psi$, which shows that $\psi : B \to C'$ is a near ring homomorphism.

To see that ψ is one-one let us assume that $(\lambda_1^{-1}x_1)\psi = (\lambda_2^{-1}x_2)\psi$. Then $[\theta_1] = [\theta_2]$ where $\theta_i : N\lambda_i \to N$ is given by $(y\lambda_i)\theta_i = y\,x_i$ $(i = 1,2)$. Now $[\theta_1] = [\theta_2]$ implies that there exists $\nu \in N\lambda_1 \cap N\lambda_2 \cap S$ such that $(\nu)\theta_1 = (\nu)\theta_2$. Since $\nu \in N\lambda_1 \cap N\lambda_2$ we have that $\nu = r_1\lambda_1 = r_2\lambda_2$ for some r_1, r_2 in N. So $r_1x_1 = (r_1\lambda_1)\theta_1 = (r_2\lambda_2)\theta_2 = r_2x_2$ which implies that $\nu\lambda_1^{-1}x_1 = \nu\lambda_2^{-1}x_2$. Since $\nu \in S$ we have that $\lambda_1^{-1}x_1 = \lambda_2^{-1}x_2$.

Lastly for any $[\tau] \in C'$ with $\tau : N\lambda \to N$ define $\theta : N\lambda \to N$ by $(x\lambda)\theta = x(\lambda)\tau$. Clearly $[\theta] = [\tau]$. Moreover $\lambda^{-1}(\lambda\tau) \in B$ and $(\lambda^{-1}(\lambda\tau))\psi = [\theta] = [\tau]$. So that ψ is an epimorphism. Thus we have seen that $\psi : B \to C'$ is a near-ring isomorphism.

Hence we have the following:

2.3. Theorem : Any two classical left near rings of left quotients of N with respect to S are isomorphic.

In view of the above theorem we have that classical left near ring C' of left quotients of N with respect to S is unique upto

isomorphism. Hence we can speak of the classical near ring of left quotients of N with respect to S.

As in the previous section, it can be proved that if N is a d.g. near ring then C' is also a d.g. near ring.

We state without proof the following:

2.4 : Theorem : If S and T are two multiplicatively closed subsets of distributive non-zero divisors of N such that $S \subseteq T$, then the classical near ring of left quotients of N with respect to S is contained in the classical near ring of left quotients of N with respect to T.

2.5. Theorem : If the classical near rings of left and right quotients of N with respect to S exist then the classical near ring of right quotients is also a classical near ring of left quotients and vice versa.

The following examples show that the concepts of classical near rings of left quotients and right quotients are quite independent of each other. That is to say that for a near ring N, classical near ring of left quotients may exist, however, classical near ring of right quotients may not exist and vice versa.

Example 1: Let R be any ring for which a ring Q_o - the classical ring of left quotients with respect to S (a semigroup of nonzero divisors of R) exists and which does not have a classical ring of right quotients.

Let $N = (R \times Q_0, +, .)$ where + and . on $R \times Q_0$ are defined as follows:

$$(r_1,q_1) + (r_2,q_2) = (r_1+r_2,\ q_1+q_2)$$

and $(r_1,q_1).(r_2.q_2) = (r_1r_2,\ q_1r_2+q_2)$ for all $r_1, r_2 \varepsilon R$ and q_1, $q_2 \varepsilon Q_0$.

It can be easily checked that N is a left near-ring and an element (r,q) of N is distributive nonzero divisor iff $q=0$ and r is a nonzero divisor in R.

Consider $\overline{S} = \{(\lambda,0) | \lambda \varepsilon S\}$, then $\overline{S}$ is a semi group of distributive nonzero divisors of N.

We claim that N has a classical near ring of left quotients with respect to $\overline{S}$. For this take $(r_1,q_1) \varepsilon N, (\lambda,0) \varepsilon \overline{S}$. Since R satisfies C.L.M.P. with respect to S, given $r_1 \varepsilon R$ and $\lambda \varepsilon S$ there exists $x' \varepsilon R$ and $\lambda' \varepsilon S$ such that $x'\lambda = \lambda' r_1$. It is easy to see that $(x', q_1\lambda^{-1})(\lambda,0) = (\lambda',0)(r_1,q_1)$, which shows that N has C.L.M.P. with respect to $\overline{S}$ and hence has a classical near ring of left quotients.

We claim that if N has a classical near ring of right quotients then R will have a classical ring of right quotients. To see this, suppose N has a classical near ring of right quotients with respect to $\overline{S}'$ - a semigroup of distributive nonzero divisors of N. Clearly elements of $\overline{S}'$ have the form $(\lambda, 0)$ where λ is a nonzero divisor in R.

We claim that R has a classical ring of right quotients with respect to $S' = \{\lambda | (\lambda,0) \varepsilon \overline{S}'\}$. For this take $r \varepsilon R$ and $\lambda \varepsilon S'$, then

$(r,\lambda^{-1}) \in N$ and $(\lambda,0) \in \overline{S}'$. Since, by assumption, N has a classical near ring of right quotients with respect to $\overline{S}'$, we have the existence of $(t_1,q_1) \in N$ and $(\lambda',0) \in \overline{S}'$ such that $(r, \lambda^{-1})(\lambda',0)= (\lambda,0)(t_1,q_1)$ that is $(r\lambda', \lambda^{-1}\lambda') = (\lambda t_1,q_1)$. Hence $r\lambda' = \lambda t_1$ which shows that R has C.R.M.P. with respect to S' and hence a classical ring of right quotients, which contradicts our assumption.

Hence N does not have a classical near ring of right quotients.

Example 2: Let R be a ring which has a classical ring of right quotients with respect to T- a semi group of nonzero- divisors of R, but does not have a classical ring of left quotients.

Let $N = (R \times R, +, .)$ where + and . are defined as follows:

$$(r_1,r_2) + (t_1,t_2) = (r_1+t_1 ,r_2+t_2)$$

and $$(r_1,r_2) \cdot (t_1,t_2) = (r_1t_1, r_2t_1+ t_2) \text{ for all } r_1,r_2,t_1,t_2 \in R.$$

Then N is a left near ring. An element $(r_1,r_2) \in N$ is a non-zero divisor and distributive iff $r_2 = 0$ and r_1 is a nonzero-divisor in R.

Set $\overline{S}= \{(\lambda,0) | \lambda \in T\}$. Then $\overline{S}$ is a semigroup of distributive nonzero divisors of N.

As in Example 1 it can be shown that N satisfies C.R.M.P. with respect to $\overline{S}$. We now prove that N satisfies condition (ii) of TheoremIII.1.5.For this let $(\lambda_1, 0)$ and $(\lambda_2,0) \in \overline{S}$. Then there exist elements $(r_1,r_2) \in N$, $(\lambda_3,0) \in \overline{S}$ such that $(\lambda_1,0)(r_1,r_2)= (\lambda_2,0)(\lambda_3,0)$

that is $(\lambda_1 r_1, r_2) = (\lambda_2\lambda_3, 0)$. So that $r_2=0$, which shows that $(r_1,r_2)= (r_1,0)$ is distributive in N. Hence N has a classical near ring of right quotients with respect to $\overline{S}$.

As in Example 1 we can show that if N has a classical near ring of left quotients then R has a classical ring of left quotients.

§ 3. Module of Quotients

Let N be a left d.g. near ring generated by the semi group of all the distributive elements of N.

Suppose that N satisfies the conditions :

(i) C.R.M.P. with respect to S and

(ii)' For every ν in S and every r in N, νr distributive in N implies that r is distributive in N.

For any right N-module M consider $M^* = \bigcup\{f: \lambda N \to M$ where $\lambda \in S$ and $f(x\mu) = f(x)\mu$ for all $x \in \lambda N$ and all distributive $\mu \in N\}$.

If $f_i \in M^*$ (i=1,2) has the domain $\lambda_i N$, define $f_1 \sim f_2$ iff there exists $\lambda \in \lambda_1 N \cap \lambda_2 N \cap S$ such that $f_1(\lambda) = f_2(\lambda)$. Clearly $\sim$ is an equivalence relation, and let $\overline{M}$ denote the set of equivalence classes of M* with respect to the above defined relation.

In $\overline{M}$ addition is defined as follows:

Let $[f_i]$ (i=1,2) be in $\overline{M}$ where $\lambda_i N$ is the domain of f_i. By C.R.M.P. there exist elements $x' \in N$ and $\lambda' \in S$ such that $\nu = \lambda_1\lambda' = \lambda_2 x' \in S \cap \lambda_1 N \cap \lambda_2 N$.

Define $\phi : \nu N \to M$ by $\phi(\nu n) = f_1(\nu n) + f_2(\nu n)$. As before it can be seen that we may write $f_1 + f_2$ for ϕ without any ambiguity. We define $[f_1] + [f_2] = [f_1 + f_2]$. It is easy to check that the addition is well defined and that $(\overline{M}, +)$ is a group.

3.1. Proposition : $\overline{M}$ is a right Q'-module.

Proof : Let $[f] \in \overline{M}$ and $[\theta] \in Q'$ and consider the mapping $\overline{M} \times Q' \to \overline{M}$ given by $([f], [\theta]) \to [f'.\theta]$ where f has the domain $\lambda_1 N$ and θ has the domain $\lambda_2 N$ and $f' : \lambda_1 N \to M$ is given by $f'(\lambda_1 n) = f(\lambda_1)n$. Also $f'.\theta : \mu N \to M$ where $\mu \in \lambda_2 N \cap S$ such that $\theta(\mu) \in \lambda_1 N$. It is a routine matter to check that the above mapping is well defined.

To show that $\overline{M}$ is a right Q'- module we give the proof of the fact that $(\overline{m}_1 + \overline{m}_2)q = \overline{m}_1 q + \overline{m}_2 q$ for all $\overline{m}_1, \overline{m}_2 \in \overline{M}$ and all distributive $q \in Q'$, the other conditions in the definition of a module can be easily verified.

We first show that if $q = [\theta]$ is a distributive element in Q' then there exists $\mu \in$ (domain of θ) $\cap S$ such that $\theta(\mu)$ is distributive in N and $q = \theta(\mu)\mu^{-1}$. Let $\theta : \lambda N \to N$ then $\theta(\lambda)\lambda^{-1} = [\psi_{\theta(\lambda)}]\ [g] = [\psi_{\theta(\lambda)} \circ g]$ where $g : \lambda N \to N$ is given by $g(\lambda n) = n$. Since $\psi_{\theta(\lambda)} \circ g(\lambda) = \theta(\lambda)$ we have $[\theta] = q = [\psi_{\theta(\lambda)} \circ g] = \theta(\lambda)\lambda^{-1}$.

For any distributive $q \in Q'$ and $n_1, n_2 \in N$ we have $([\psi_{n_1}] + [\psi_{n_2}])\ [\theta] = [\psi_{n_1}][\theta] + [\psi_{n_2}][\theta]$ where $q = [\theta]$. Hence $[\psi_{n_1+n_2} \circ \theta] = [\psi_{n_1} \circ \theta + \psi_{n_2} \circ \theta]$, which implies that there exists

$\mu \varepsilon$ (domain of θ) $\cap$ S such that $\psi_{n_1+n_2} \circ \theta(\mu) = (\psi_{n_1} \circ \theta + \psi_{n_2} \circ \theta)(\mu)$, which gives that $(n_1+n_2)\,\theta(\mu) = n_1\theta(\mu) + n_2\theta(\mu)$, that is $\theta(\mu)$ is distributive in N. Now $q = [\theta] = \theta(\lambda)\,\lambda^{-1} = \theta(\lambda)\, s\mu^{-1} = \theta(\lambda s)\mu^{-1}$ $= \theta(\mu)\,\mu^{-1}$ where $\mu = \lambda s \;\varepsilon\; S$ for some distributive s in N since $\mu \;\varepsilon\; \lambda N \cap S$, λN being the domain of θ.

Now let $\overline{m}_1 = [f_1]$, $\overline{m}_2 = [f_2]$ be in $\overline{M}$ and $q = [\theta]$ be a distributive element of Q'. Suppose that $\lambda_i N$ ($i = 1,2$) is the domain of f_i and λN is the domain of θ. Given λ_1, λ_2 there exists $x' \;\varepsilon\; N$ and $\lambda' \;\varepsilon\; S$ such that $\nu = \lambda_1 x' = \lambda_2 \lambda'$, so that $f_1 + f_2 : \nu N \to M$ and $(f_1 + f_2)' \circ \theta : bN \to M$ where $b \;\varepsilon\; \lambda N \cap S$ such that $\theta(b) \;\varepsilon\; \nu N$ say $\theta(b) = \nu x$. Then $(\overline{m}_1 + \overline{m}_2)q = ([f_1] + [f_2])\,[\theta] = [(f_1 + f_2)' \circ \theta]$ and $\overline{m}_1 q + \overline{m}_2 q = [f_1]\,[\theta] + [f_2]\,[\theta] = [f_1' \circ \theta + f_2' \circ \theta]$.

Let $a \;\varepsilon\; bN \cap \mu N \cap S$ where $\mu \;\varepsilon\; \lambda N \cap S$ is such that $\theta(\mu)$ is distributive in N, then $a = bn_1 = \mu n_2 \;\varepsilon\; S$ and by hypothesis n_1, n_2 are distributive. Now $\theta(a) = \theta(\mu)n_2 = \theta(b)n_1 = \nu x.n_1$. Since $\theta(\mu)$ and n_2 both are distributive in N, $\theta(a)$ is distributive and hence $xn_1 = \nu^{-1}\,\theta(a)$ is a distributive element of N.

Now $(f_1+f_2)' \circ \theta(a) = (f_1' \circ \theta + f_2' \circ \theta)(a)$, so that $(\overline{m}_1+\overline{m}_2)q$ $= \overline{m}_1 q + \overline{m}_2 q$. Hence $\overline{M}$ is a right Q'-module.

Now let, for any $m \;\varepsilon\; M$ $\phi_m : N \to M$ be given by $\phi_m(n) = mn$. Then $[\phi_m] \;\varepsilon\; \overline{M}$ and $\pi_M : M \to \overline{M}$, given by $\pi_M(m) = [\phi_m]$, maps M into $\overline{M}$ and is an N-homomorphism. Moreover $\ker \pi_M = \{m \;\varepsilon\; M \mid [\phi_m] = 0\}$ $= \{m \;\varepsilon\; M \mid m\lambda = 0$ for some $\lambda \;\varepsilon\; S\}$. So $M \subseteq \overline{M}$ iff no element of M

is annihilated by an element of S.

Now let M_1 and M_2 be two right N-modules and $\phi : M_1 \to M_2$ be a N-homomorphism. Consider $\bar{\phi} : \bar{M}_1 \to \bar{M}_2$ given by $\bar{\phi}([f]) = [\phi \circ f]$ for all $[f] \in \bar{M}_1$. It can be checked that $\bar{\phi}$ is a Q'- homomorphism. Clearly $\bar{\phi} \circ \pi_{M_1} = \pi_{M_2} \circ \phi$ and $\bar{\phi}$ is uniquely determined by ϕ. Moreover the association of $\bar{M}$ with M and the association of $\bar{\phi}$ with ϕ is a covariant functor F from the category of right N-modules and their homomorphisms into the category of Q'-modules and their homomorphisms.

It is easy to see that F is left exact. Thus we have proved

3.2. Proposition : If $\phi : M_1 \to M_2$ is an N-homomorphism, then ϕ induces a unique Q'-module homomorphism $\bar{\phi} : \bar{M}_1 \to \bar{M}_2$ such that $\bar{\phi} \circ \pi_{M_1} = \pi_{M_2} \circ \phi$. Finally if $0 \to M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3$ is an exact sequence of N-modules then $0 \to \bar{M}_1 \xrightarrow{\bar{\phi}} \bar{M}_2 \xrightarrow{\bar{\psi}} \bar{M}_3$ is an exact sequence of Q'-modules.

3.3. Definition : An N-module M is called S-free iff $ms = 0$, $m \in M$, $s \in S$ implies $m = 0$.

If $T = \{m \in M \mid \text{there exists } \lambda \in S \text{ such that } m\lambda = 0\}$, then T is a N- submodule of M and M/T is a S-free N-module.

3.4. Proposition : For any N-module M, $\bar{M}$ is a S-free N-module.

Proof : Since $\bar{M}$ is a Q'-module, it is a N-module. Take $\bar{m} = [f]$ in $\bar{M}$ and suppose that $[f]\lambda = 0$ for some $\lambda \in S$. So we have

$[f][\psi_\lambda] = [f' \circ \psi_\lambda] = 0$, which implies that $f' \circ \psi_\lambda(\mu) = 0$ for some $\mu \in (\mathrm{dom}\, f' \circ \psi_\lambda) \cap S$. Hence $f'(\lambda\mu) = 0$ which implies that $[f'] = [f] = \overline{m} = 0$.

3.5. Proposition : Let $\nu : M \longrightarrow M/T$ be the natural N-epimorphism. Then $\bar{\nu} : \overline{M} \to \overline{\left(\frac{M}{T}\right)}$ is one-one.

Proof : Let $[f] \in \overline{M}$ such that $\bar{\nu}([f]) = [\nu \circ f] = 0$. Then $\nu \circ f(\lambda) = 0$ for some $\lambda \in (\mathrm{dom}\, \nu \circ f) \cap S$ which implies that $\nu(f(\lambda)) = f(\lambda) + T = 0$, which again implies that $f(\lambda)\mu = f(\lambda\mu) = 0$ for some $\mu \in S$. Hence $[f] = 0$.

3.6. Lemma : Let M be a S-free N-module and μN be a regular N-subgroup. Then for any map $f: \mu N \to \overline{M}$, having the property $f(xn) = f(x)n$ for all $x \in \mu N$ and all $n \in N$, there exists a map $h: N \to \overline{M}$ such that $[f] = [h]$ and h also satisfies $h(nn') = h(n)\, n'$ for all $n, n' \in N$.

Proof : Let $E = \{x \in \mu N \mid f(x) \in \pi_M(M)\}$. Since $f(\mu) \in \overline{M}$, $f(\mu) = [\theta]$ where $\theta : \lambda N \to M$ for some $\lambda \in S$. For all $n \in N$, $f(\mu\lambda n) = f(\mu)\lambda n = [\theta][\psi_{\lambda n}] = [\theta' \circ \psi_{\lambda n}] = [\psi_{\theta'(\lambda n)}] = \pi_M(\theta'(\lambda n)) \in \pi_M(M)$. Hence $\mu\lambda n \in E$ for all $n \in N$, so that $\mu\lambda N \subsetneq E$. So that for all $x \in \mu\lambda N$, $f(x) \in \pi_M(M)$ and therefore $f(x) = \pi_M(m_x) = [\psi_{m_x}]$ for some $m_x \in M$. Define $g: \mu\lambda N \to M$ by $g(x) = m_x$ for all $x \in \mu\lambda N$. Now $x_1 = x_2 \in \mu\lambda N \Rightarrow f(x_1) = f(x_2)$ i.e. $[\psi_{m_{x_1}}] = [\psi_{m_{x_2}}]$, or $\psi_{m_{x_1}}(s) = \psi_{m_{x_2}}(s)$ for some $s \in S$, hence $m_{x_1} s = m_{x_2} s$, or $(m_{x_1} - m_{x_2})s = 0$, so $m_{x_1} = m_{x_2}$ since M is S-free. Hence g is well defined.

We now show that $g(xn') = g(x)n'$ for all $x \in \mu\lambda N$ and all $n' \in N$. Now $g(xn') = m_{xn'}$ and $g(x)n' = m_x n'$. Also $f(xn') = [\psi_{m_{xn'}}]$ $= f(x)n' = [\psi_{m_x}][\psi_{n'}] = [\psi_{m_x} \circ \psi_{n'}]$, so that there exists $s' \in S$ such that $\psi_{m_{xn'}}(s') = \psi_{m_x} \cdot \psi_{n'}(s')$ which implies $m_{xn'}s' = \psi_{m_x}(n's') = m_x n's'$, that is $m_{xn'} = m_x n'$.

Define $h: N \to \overline{M}$ by $h(x) = [g]\, x$ then h is well defined and $h(nn') = h(n)n'$ for all $n, n' \in N$. We now show that $[h] = [f]$. For any $x \in \mu\lambda N$ we have $h(x) = [g]\, x = [g \circ \psi_x]$ and $f(x) = [\psi_{m_x}]$, and for any $y \in (\text{dom } g \circ \psi_x) \cap (\text{dom } \psi_{m_x})$ we have $\psi_{m_x}(y) = m_x y$ and $g \circ \psi_x(y) = g(xy) = g(x)y = m_x y$ and so $[h] = [f]$.

3.7. Proposition : If M is S-free module then $\pi_{\overline{M}} : \overline{M} \to \overline{\overline{M}}$ is an isomorphism.

Proof : Since M is S-free, $\ker \pi_{\overline{M}} = 0$.

Take any $[f] \in \overline{\overline{M}}$, then $f: \lambda N \to \overline{M}$. Define $f' : \lambda N \to \overline{M}$ by $f'(\lambda n) = f(\lambda)n$, then by the lemma 3.6, there exists $h: N \to \overline{M}$ such that $[h] = [f']$. We claim that $[f] = [f'] = \pi_{\overline{M}}(h(1))$.

Now $\pi_{\overline{M}}(h(1)) = [\psi_{h(1)}]$ where $\psi_{h(1)} : N \to \overline{M}$. $[f'] = [h]$ implies that there exists $\lambda n \in \lambda N \cap S$ with n distributive in N such that $h(\lambda n) = f'(\lambda n) = f(\lambda)n = f(\lambda n)$, that is $\psi_{h(1)}(\lambda n) = h(1)\lambda n = h(\lambda n) = f(\lambda n)$ so that $[\psi_{h(1)}] = [f]$ which implies that $\pi_{\overline{M}}$ is epimorphism. Hence the result.

3.8. Theorem : Let M be a module over Q'. Then M is clearly a module over N. M is an injective Q'-module iff M is an injective N-module.

Proof : Since the proof is same as the proof of Theorem 2.1 [9] we will only outline the proof here. Suppose M is an injective Q'-module. Let $f: I_N \to M_N$ be a N-homomorphism where I is a right ideal of N. Now $IQ' = \{as^{-1} \mid a \in I, s \in S\}$ is a right ideal of Q'. Define $f' : IQ' \to M$ by $f'(as^{-1}) = f(a)s^{-1}$ then f' is a Q'-homomorphism. M being an injective Q'-module there exists $m \in M$ such that $f'(x) = mx$ for all $x \in IQ'$. But $f'(x) = f(x)$ for all $x \in I$. Therefore $f(x) = mx$ for all $x \in I$.

Conversely suppose M_N is injective. Let $f: I \to M$, be Q' homomorphism, where I is a right ideal of Q'. Then $J = I \cap N$ is a right ideal of N and $JQ' = \{x s^{-1} / x \in J, s \in S\} = I$.

Let f' denote the restriction of f to J. Then f' is clearly a N-homomorphism of J into M. As M is an injective N-module, there exists $m \in M$ such that $f'(x) = mx$ for all $x \in J$. Any element of I is of the form $x s^{-1}$, $x \in J$, $s \in S$, $f(x s^{-1}) = f'(x)s^{-1} = m(xs^{-1})$. Hence M is an injective Q' - module.

3.9. Theorem : Let M be a S-free module over N. Then $\overline{M}$ is an injective Q' module (or injective N-module) iff every N-homomorphism $f: I \to M$, I-a right ideal of N, can be extended to an N-homomorphism

$g: J \to M$, where J is a right ideal of N containing I and containing an element of S.

Proof : Let M_N satisfy the condition. To show $\overline{M}$ is injective, consider $f: I \to \overline{M}$ a N-homomorphism where I is a right ideal of N.

Let $I' = \{x \in I \mid f(x) \in M\}$, then I' is an N-subgroup. Let u be the ideal of N generated by I', then $u = \{\sum_{i=1}^{n} -n_i + x_i + n_i \mid n_i \in N, x_i \in I'\}$. Define $f': u \to M$ by $f'(\sum_{i=1}^{n} -n_i + x_i + n_i) = \sum_{i=1}^{n} f(x_i)$. It is easy to see that f' is a well defined N-homomorphism. Hence by hypothesis there exists $g: J \to M$ where J is a right ideal of N containing u and containing an element of S and $g(x) = f'(x)$ for all $x \in u$.

Consider $JQ' = \{xs^{-1} \mid x \in J,\ s \in S\}$ then JQ' is a right ideal of Q'. Define $g': JQ' \to \overline{M}$ by $g'(xs^{-1}) = g(x)s^{-1}$, then g' is a Q'-homomorphism. Since J contains an element of S say λ , we have $\lambda\lambda^{-1} = 1 \in JQ'$. Hence $JQ' = Q'$. Let $g'(1) = \overline{m} \in \overline{M}$. We claim that for each $r \in I$ there exists $s \in S$ such that $rs \in I'$. To this end take $r \in I$, then $f(r) \in \overline{M}$, say $f(r) = [\theta]$ where θ has the domain μN. Now $f(r)\mu = f(r\mu) = [\theta]\,[\psi_\mu] = [\theta' \circ \psi_\mu] = [\psi_{\theta(\mu)}] = \theta(\mu) \in M$, so that $r\mu \in I'$ and hence for all $r \in I$, $f(r)\mu = f(r\mu) = f'(r\mu) = g(r\mu) = g'(r\mu) = g'(1)\,r\mu = \overline{m}(r\mu) = (\overline{m}r)\mu$ which gives that $f(r) = \overline{m}\,r$ for all $r \in I$.

Conversely, suppose that $\overline{M}$ is an injective N-module (injective Q'-module). Consider an N-homomorphism $f: I \to M$ where

I is a right ideal of N. Since $\overline{M}$ is injective, there exists $\overline{m} \in \overline{M}$ such that $\pi_M \circ f(x) = \overline{m}x$ for all $x \in I$, that is $\pi_M(f(x)) = [\psi_{f(x)}] = \overline{m}x$. Since M is S-free, $M \subseteq \overline{M}$ and since $f(x) \in M$ we have $[\psi_{f(x)}] = f(x) = \overline{m}x \in M$ for all $x \in I$. Now $\overline{m} \in \overline{M} \Longrightarrow \overline{m} = [\theta]$ with $\theta : \lambda N \to M$ and so $\overline{m} = \theta(\lambda)\lambda^{-1}$. Hence $f(x) = \theta(\lambda)\lambda^{-1}x$ for all $x \in I$.

Let $J = \lambda N + I$ then J is a right N-subgroup. Define $g: J \to M$ by $g(\lambda r + x) = \theta(\lambda)\lambda^{-1}(\lambda r + x) = \theta(\lambda)r + \theta(\lambda)\lambda^{-1}x$ for all $r \in N$ and all $x \in I$. Clearly g is well defined and $g|_I = f$.

We now show that g is an N-homomorphism. We have

$$
\begin{aligned}
g\{(\lambda r_1+x_1)+(\lambda r_2+x_2)\} &= g\{(\lambda r_1+\lambda r_2)+(-\lambda r_2+x_1+\lambda r_2+x_2)\} \\
&= g\{\lambda(r_1+r_2) + (-\lambda r_2+x_1+\lambda r_2)+ x_2\} \\
&= \theta(\lambda)\lambda^{-1}(\lambda(r_1+r_2))+\theta(\lambda)\lambda^{-1}(-\lambda r_2+x_1+\lambda r_2+x_2) \\
&= \theta(\lambda)(r_1+r_2)+ \theta(\lambda)(-r_2)+ \theta(\lambda)\lambda^{-1}x_1+ \theta(\lambda)r_2+\theta(\lambda)\lambda^{-1}x_2 \\
&= (\theta(\lambda)r_1+ \theta(\lambda)\lambda^{-1}x_1) + (\theta(\lambda)r_2 +\theta(\lambda)\lambda^{-1}x_2) \\
&= g(\lambda r_1+x_1) + g(\lambda r_2+x_2).
\end{aligned}
$$

It is easy to see that $g\{(\lambda r+x)s\} = g(\lambda r+x)s$ where s is any element of N such that either s or $-s$ is distributive. Since N is a d.g. near ring we have that g is an N-homomorphism.

Define $g': B = \{\sum_{i=1}^{n} -n_i+x_i+n_i \mid n_i \in N,\ x_i \in J\} \to M$ by $g'(\sum_{i=1}^{n} -n_i+x_i+n_i) = \sum_{i=1}^{n} g(x_i)$. It can be easily checked that g'

is an N-homomorphism and $g'|_J = g$.

Now B is a right ideal of N such that $I \subseteq J \subseteq B$. Since J contains an element of S, B contains an element of S and $g'|_I = f$. Hence the result.

3.10. Corollary : Q'_N is injective iff for every N-homomorphism $f:I \to N$, I a right ideal of N, there exists an N-homomorphism $g:J \to N$, where J is a right ideal of N containing I and containing an element of S and g is such that $g(x) = f(x)$ for all $x \in I$.

CHAPTER IV

COMPLETE NEAR RING OF QUOTIENTS

This chapter is devoted to the study of complete near rings of left and right quotients. It is proved that if Q denotes the complete near ring of right quotients of N, then Q is a regular near ring whenever

$$J(N_N) = \{ \sum_{j=1}^{n} -r_j + x_j s_j + r_j \mid \text{there exist weakly large } N\text{-subgroups } K_j \text{ such that } x_j K_j = 0, x_j, r_j \in N, s_j \in \text{generating set of } N\} = 0$$

Moreover it is shown that if every weakly large right N-subgroup of N_N contains a distributive nonzero divisor then Q is a classical near ring of right quotients of N.

§1. Complete Near Ring of Right Quotients

Let N be a left d.g. near ring with identity and I_N be the injective hull of N_N.

Let $H = \{f : I \to I \mid (is)f = (i)fs$ for all distributive s in N$\}$., then H is a left near ring. Define $H \times I \to I$ by $(h, i) \to h.i$ where $h.i = (i)h$, so that $HI \subseteq I$.

Let $Q = \{\theta : I \to I \mid (h.i)\theta = h.(i)\theta = ((i)\theta)h$ for all N-homomorphism h in H$\}$, then Q is a left near ring.

1.1. Lemma : $\phi : N \to Q$ given by $\phi(n) = \phi_n$ is a monomorphism , where

$\phi_n : I \to I$ is such that $(i)\phi_n = in$.

Proof : For any right N-homomorphism h in H we have $(h.i)\phi_n = ((i)h)\phi_n = (i)hn = (in)h = h.((i)\phi_n)$ and hence ϕ_n belongs to Q.

Now (i) $\phi_{n_1+n_2} = i(n_1+n_2) = in_1 + in_2 = (i)\ \phi_{n_1} + (i)\phi_{n_2}$ implies

$$\phi_{n_1+n_2} = \phi_{n_1} + \phi_{n_2}\ .\ \text{Hence } \phi \text{ is additive .}$$

Also (i) $\phi_{n_1 n_2} = i(n_1 n_2) = (in_1)n_2 = (in_1)\phi_{n_2} = ((i)\phi_{n_1})\phi_{n_2} = (i)\phi_{n_1} \circ \phi_{n_2}$

implies $\phi(n_1 n_2) = \phi(n_1)\ \phi(n_2)$.

To see that ϕ is one-one, suppose $\phi(n) = \phi_n = 0$, this implies that $(1)\phi_n = 0 = 1.n = n$.

Hence we can say that N is contained in Q.

1.2. Lemma : For any i in I there exists a right N-homomorphism h in H such that $i = (1)h$.

Proof : For any i in I consider $\psi_i : N_N \to I_N$ given by $(n)\psi_i = in$, then ψ_i is a N-homomorphism. By injectivity of I, ψ_i may be extended to some N-homomorphism $h: I_N \to I_N$. Thus we have a h in H such that $(1)h = (1)\psi_i = i$.

1.3. Lemma : $\phi : Q_N \to I_N$ given by $\phi(q) = (1)q$, is a monomorphism.

Proof : Now $\phi(q_1+q_2) = (1)\ (q_1+q_2) = (1)q_1 + (1)q_2 = \phi(q_1) + \phi(q_2)$ and $\phi(qn) = (1)(qn) = (1)qn = \phi(q)n$ implies that ϕ is a right N-homomorphism.

Now suppose $\phi(q) = 0$, this implies $(1)q = 0$. For any i in I, $(i)q = ((1)h)q = (h.1)q = h((1)q = ((1)q)h = 0$ where h is a right N-homomorphism in H. So that $q = 0$. Hence ϕ is a monomorphism.

We shall denote the image of Q in I by 1.Q.

1.4. Proposition : $1.Q = \{i \varepsilon I \mid$ for any pair of homomorphisms h_1, h_2 in H, $(n)h_1 = (n)h_2$ for all $n \varepsilon N$ implies $(i)h_1 = (i)h_2\}$

Proof : Let h_1, h_2 be homomorphisms in H such that $(n)h_1 = (n)h_2$ for all n in N. For any $(1)q$ in 1.Q we have $((1)q)h_1 = ((1)h_1)q = ((1)h_2)q = ((1)q)h_2$ which implies $(1)q$ belongs to the given N-subgroup of I_N. Conversely let i belong to I such that for all homomorphisms h_1, h_2 in H and all n in N , $(n)h_1 = (n)h_2$ implies $(i)h_1 = (i)h_2$. Consider $\phi : I \to I$ such that $((1)h)\phi = (i)h$ where h is a N-homomorphism in H. Now $i_1 = i_2$ implies $(1)h_1 = (1)h_2$ so that $(n)h_1 = (n)h_2$ for all n in N. So that $(i)h_1 = (i)h_2$. Hence ϕ is well defined. Also $(h.i')\phi = ((i')h)\phi = (((1)h')h)\phi = ((1)h'h)\phi = (i)h'h$

$$= ((i)h')h = ((1)h'\phi)h = ((i')\phi)h = h.(i')\phi .$$

$$(i' = (1)h', \ h' \text{ is a homomorphism in } H)$$

So that ϕ belongs to Q. Taking $\phi = q$ we have that there exists q in Q such that $((1)h)q = (i)h$. Take h = identity map then $i = (1)q$ and this implies that the given N-subgroup of I_N is contained in 1.Q. Hence the result.

1.5. Definition : A right N-subgroup D_N of I_N is called strictly-dense iff for any pair of homomorphisms $h_1, h2$ in H, $(d)h_1 = (d)h_2$ for all d

in D implies $(n)h_1 = (n)h_2$ for all n in N.

1.6. Trivially N_N is strictly-dense.

1.7. Lemma : If a right N-subgroup D_N of Q_N is strictly-dense, then for any q in Q, $q^{-1}D = \{n \in N \mid q.n \in D\}$ is also strictly-dense.

Proof : Let h_1, h_2 be homomorphisms in H such that $(n)h_1 = (n)h_2$ for all n in $q^{-1}D$. Define $\phi : B = D + qN \to I$ by $(d+qr)\phi = (r)(-h_1+h_2)$ for all d in D and all r in N. Now $d_1 + qr_1 = d_2+qr_2$ implies $-d_2+d_1 = qr_2-qr_1 = q(r_2-r_1) \in D$. So that $r_2-r_1 \in q^{-1}D$. Hence $(r_2-r_1)h_1 = (r_2-r_1)h_2$, or $(r_2)h_1+(-r_1)h_1 = (r_2)h_2+(-r_1)h_2$ that is $(r_2)h_1+(r_1)(-h_1) = (r_2)h_2+(r_1)(-h_2)$, so that $(r_1)(-h_1+h_2) = (r_2)(-h_1+h_2)$. Hence ϕ is well defined.

$$\text{Let } \overline{B} = \{\sum_{i=1}^{n}(d_i+qr_i)s_i \mid (d_i+qr_i) \in B,\ s_i \text{ or } -s_i \text{ belongs to the generating set of } N\}$$

Then $\overline{B}$ is a right N module contained in Q and hence in I. Define $\bar{\phi} : \overline{B} \to I$ by $(\sum_{i=1}^{n}(d_i+qr_i)s_i)\bar{\phi} = \sum_{i=1}^{n}(d_i+qr_i)\phi\, s_i$. We note that $\bar{\phi}$ is additive and coincides with ϕ on B. We now prove that $\bar{\phi}$ vanishes on the zero element of $\overline{B}$. To this end, let $\sum_{i=1}^{n}(d_i+qr_i)s_i = 0$, this implies that

$$(d_1+qr_1)s_1 + (d_2+qr_2)s_2 + \dots + (d_{n-1}+qr_{n-1})s_{n-1} + (d_n+qr_n)s_n = 0$$

$$\text{or } (d_1+qr_1)s_1 + (d_2+qr_2)s_2 + \dots + (d_{n-1}+qr_{n-1})s_{n-1} = -(d_n+qr_n)s_n$$

Case I : If $-s_n$ is distributive then we have

$$(d_1+qr_1)s_1 + (d_2+qr_2)s_2 + \dots + (d_{n-1}+qr_{n-1})s_{n-1} = (d_n+qr_n)(-s_n)$$
$$= d_n(-s_n) + qr_n(-s_n) \in D + qN.$$

Hence $\quad (L.H.S.)\phi = \{d_n(-s_n) + q\, r_n(-s_n)\}\phi$

or $\quad (L.H.S.)\bar{\phi} = \{d_n(-s_n) + qr_n(-s_n)\}\phi$

or $\quad (d_1+qr_1)\phi s_1 + (d_2+qr_2)\phi s_2 + \ldots\ldots + (d_{n-1}+qr_{n-1})\phi\ s_{n-1} = (-r_n s_n)(-h_1+h_2)$

or $\quad \sum_{i=1}^{n-1} (r_i)(-h_1+h_2)s_i = (r_n)(-h_1+h_2)(-s_n) = -(r_n)(-h_1+h_2)s_n$

or $\quad \sum_{i=1}^{n} (r_i)(-h_1+h_2)s_i = 0$ which implies that

$$\sum_{i=1}^{n} (d_i + qr_i)\phi\, s_i = 0 = \left(\sum_{i=1}^{n}(d_i+qr_i)s_i\right)\bar{\phi} .$$

Case II. If s_n is distributive then we have

$$-(d_{n-1}+qr_{n-1})s_{n-1} - \ldots - (d_2+qr_2)s_2 - (d_1+qr_1)s_1 = (d_n + qr_n)s_n$$

$$= d_n s_n + qr_n s_n \in D + qN.$$

Hence $\quad \{(L.H.S.)\}\phi = (d_n s_n + qr_n s_n)\phi$

or $\quad (L.H.S.)\bar{\phi} = (d_n + qr_n)\phi\ s_n$

or $\quad (d_{n-1}+qr_{n-1})\phi\ (-s_{n-1}) + (d_{n-2}+qr_{n-2})\phi\ (-s_{n-2}) + \ldots + (d_1+qr_1)\phi(-s_1)$

$$= (d_n+qr_n)\phi\ s_n$$

or $\quad -(d_{n-1}+qr_{n-1})\phi\ s_{n-1} - \ldots - (d_1+qr_1)\phi\ s_1 = (d_n+qr_n)\phi\ s_n$

or $\quad (d_1+qr_1)\phi\ s_1 + (d_2+qr_2)\phi\ s_2 + \ldots + (d_{n-1}+qr_{n-1})\phi\ s_{n-1} = -(d_n+qr_n)\phi\ s_n$

which gives that $\sum_{i=1}^{n}(d_i+qr_i)\phi\ s_i = 0 = \left(\sum_{i=1}^{n}(d_i+qr_i)s_i\right)\bar{\phi}$.

Hence $\bar{\phi}$ is well defined. It is easy to see that $\bar{\phi}$ is an N-homomorphism. By injectivity of I_N, $\bar{\phi}$ can be extended to an N-homomorphism $h' : I_N \to I_N$ such that $h'|_{\bar{B}} = \bar{\phi}$.

So that $(d)h' = (d)\phi$ for all d in D. Hence $(d)h' = (0)(-h_1+h_2) = 0$ which implies that $(1)h' = 0$, since D is strictly dense. Now $(1)(-h_1+h_2) = (q)\phi = (q)h' = ((1)q)h' = (1)h'q = 0$ implies $(1)\, h_1 = (1)h_2$. Hence $(n)h_1 = (n)h_2$ for all n in N. So that q^{-1} D is strictly dense.

1.8. Definition : A right N-subgroup $D_N \subseteq N_N$ is dense iff for all $0 \neq r_1 \in N$ and all $r_2 \in N$, there exists $r \in N$ such that $r_1 r \neq 0$ and $r_2 r \in D$.

1.9. Proposition : A right N-subgroup $D_N \subseteq N_N$ is strictly dense implies D_N is dense.

Proof : Let D be strictly dense. Let $r_1 \neq 0$, $r_2 \in N$ then $r_2^{-1}D$ is strictly dense. Consider $\phi : N \to N$ such that $(r)\phi = r_1 r$. Extend ϕ to a homomorphism $h : I_N \to I_N$, then $r_1(r_2^{-1} D) = 0$ implies $(r_2^{-1} D)h=0$. Hence $(1)h = 0$, so that $(1)\ \phi = r_1 = 0$, a contradiction. Hence $r_1(r_2^{-1}D) \neq 0$ which implies there exists r in $r_2^{-1}D$ such that $r_1 r \neq 0$ and $r_2 r$ is in D. Hence D is dense.

1.10. Definition : A left near ring $S \supseteq N$ is a left near ring of right quotients of N iff for all $0 \neq s_1$ in S, and all s_2 in S, $s_1(s_2^{-1}N) \neq 0$ and $s_2^{-1}N$ is strictly dense.

Remark : Let S be a left near ring of right quotients of N and T be a left near ring such that $N \subseteq T \subseteq S$, then S is a near ring of right

quotients of T.

1.11. Proposition : Q is a left near ring of right quotients of N and it contains any left near ring of right quotients of N.

Proof : Consider $q_1 \neq 0$, q_2 in Q. Since N is strictly dense, $q_2^{-1}N$ is strictly dense. Now $0 \neq q_1 : I \to I$ implies there exists i in I such that $(i)\,q_1 \neq 0$, that is $((1)h)q_1 \neq 0$ for some N-homomorphism h in H. Hence $((1)q_1)h \neq 0$ which implies $(1)q_1 \neq 0$, that is $(1)h' \neq 0$ where $(1)q_1 = (1)h'$ for some N-homomorphism h' in H.

Since q_2^{-1} N is strictly dense, $(1)h'(q_2^{-1}N) \neq 0$ which implies there exists r in q_2^{-1} N such that $(1)h'r = (1)q_1r \neq 0$. Hence $(1)(q_1.r) \neq 0$ implies $q_1.r \neq 0$, that is $q_1.(q_2^{-1}N) \neq 0$.

Now let S be any left near ring of right quotients of N. Let $0 \neq U$ be a submodule of S_N, then there exists $0 \neq u$ in $U_N \subseteq S_N$ such that $u(u^{-1}N) = uN \cap N \neq 0$. This implies $U_N \cap N_N \neq 0$, that is S_N is a weakly essential extension of N_N and hence $S_N \subseteq I_N$.

We now claim that $S \subseteq 1.Q$. Let h_1, h_2 be two homomorphisms in H such that $(r)h_1 = (r)h_2$ for all r in N. For any s in S, we have $(s)h_1r = (s)h_2r$ for all r in $s^{-1}N$, that is $(1)h_1'\,r = (1)h_2'r$ where $(s)h_i = (1)h_i'$ $(i=1,2)$. Hence $(r)h_1' = (r)h_2'$ for all r in $s^{-1}N$ (strictly dense). So that $(1)h_1' = (1)h_2'$ that is $(s)h_1 = (s)h_2$. Hence $S \subseteq 1.Q \simeq Q$.

We now show that S is contained in Q as a near ring. For this define $\phi : S \to Q$ by $\phi(s) = q$ where $q : I \to I$ such that $((1)h)q = (s)h$.

Now $(1)h_1 = (1)h_2$ for some N-homomorphisms h_1, h_2 in H implies $(r)h_1 = (r)h_2$ for all r in N, so that $(s)h_1 = (s)h_2$ since $S \subseteq 1.Q$. Hence q is well defined.

Now $s_1 = s_2$ be in S, and let $\phi(s_1) = q_1$, $\phi(s_2) = q_2$ then for all $i \in I$, $(i)q_1 = ((1)h)q_1 = (s_1)h = (s_2)h = ((1)h)q_2 = (i)q_2$ implies ϕ is well defined.

It is easy to see that ϕ is additive. To prove that ϕ is a near ring homomorphism, we first note that $\phi(sr) = \phi(s)r$ for all s in S and all r in N. Now let s_1, s_2 be any two elements of S and suppose $\phi(s_1 s_2 r) = q$ for any r in $s_2^{-1}N$. Moreover, let $\phi(s_1) = q_1$, $\phi(s_2 r) = q_2$. Then

$$((1)h)q = (s_1 s_2 r)h = (s_1)h\ s_2 r = (1)h' s_2 r (\text{where } (s_1)h = (1)h' \in I)$$

$$= (s_2 r)h' = ((1)h')q_2 = ((s_1)h)q_2 = (((1)h)q_1)q_2 = ((1)h)q_1 q_2.$$

So $\phi(s_1 s_2 r) = \phi(s_1)\ \phi(s_2 r)$ that is $\phi(s_1 s_2)\ r = \phi(s_1)\phi(s_2)\ r$ for all $r \in s_2^{-1}N$.

Therefore $(1)h_1 r = (1)h_2 r$ for all $r \in s_2^{-1}N$ where $\phi(s_1 s_2) = (1)h_1 \in Q \subseteq I$ and $\phi(s_1)\ \phi(s_2) = (1)h_2 \in Q \subseteq I$.

Hence $(1)h_1 = (1)h_2$ and so $\phi(s_1 s_2) = \phi(s_1)\ \phi(s_2)$.

Now let $\phi(s_1) = \phi(s_2)$, then $q_1 = q_2$ where $((1)h)q_1 = (s_1)h$ and $((1)h)q_2 = (s_2)h$. Then $((1)\underline{1})q_1 = ((1)\underline{1})q_2$, so that $(s_1)\underline{1} = (s_2)\underline{1}$, hence $s_1 = s_2$ where $\underline{1}$ denotes the identity map from I to I. Hence ϕ is a monomorphism.

We would like to know what happens if we go through the same procedure again, starting with Q_Q in place of N_N ?

If we define $I \times Q \to I$ by $(i,q) \to i.q = (i)q$ then I becomes a right Q module and we have the following :

1.12. Proposition : I_Q is the injective hull of the canonical image of Q_Q , and $\mathrm{Hom}_Q(I, I) = \mathrm{Hom}_N(I,I)$.

Proof : Consider
$$\begin{array}{ccccc} 0 & \to & A_Q & \xrightarrow{f} & B_Q \\ & & \downarrow \phi & & \\ & & I_Q & & \end{array}$$
where ϕ and f are Q-homomorphisms and $0 \to A_Q \xrightarrow{f} B_Q$ exact. Since I_N is injective, there exists a N-homomorphism $\psi : B_Q \to I_Q$ which makes the above diagram commute. I_Q will be injective if we show that ψ is a Q-homomorphism.

Let $q \in Q$ arbitrary and b in B be arbitrary. Now $r \in q^{-1}N$ implies $qr \in N$. Then $\psi(bqr) = \psi(b)qr$, also $\psi(bqr) = \psi(bq)r$. Thus for all r in $q^{-1}N$, $\psi(b)qr = \psi(bq)r$. That is $i_1 r = i_2 r$ where $i_1 = \psi(b)q$, $i_2 = \psi(bq) \in I$, so that $(1)h_1 r = (1)h_2 r$, that is $(r)h_1 = (r)h_2$ for all r in q^{-1} N. Hence $(1)h_1 = (1)h_2$, that is $i_1 = i_2$, so that $\psi(b)q = \psi(bq)$. Hence ψ is a Q-homomorphism, as required.

Since I_N is weakly essential extension of N_N, it is also a weakly essential extension of $(1.Q)_N$. Therefore I_Q is weakly essential extension of $(1.Q)_Q$. Being injective it is the injective hull of $(1.Q)_Q$.

Obviously, $\mathrm{Hom}_Q(I,I) \subseteq \mathrm{Hom}_N(I,I)$. Now let $\psi : I \to I$ be a N-homomorphism and let q in Q, $i \in I$ be arbitrary. Then $(iqr)\psi = (iq)\psi r = (i)\psi qr$ for all r in $q^{-1}N$ implies that $i_1 r = i_2 r$, that is $(1)h_1 r = (1)h_2 r$ for all r in $q^{-1}N$ (strictly dense) where $i_1 = (iq)\psi = (1)h_1$ and $i_2 = (i)\psi q = (1)h_2$. Hence $(1)h_1 = (1)h_2$, which shows that ψ is a Q-homomorphism. Hence the result.

1.13. Corollary: Q is its own near ring of right quotients.

Proof : Let $H' = \{ f : I_Q \to I_Q \mid (iq)f = (i)fq$ for all distributive q in Q and all i in $I \}$

and $\overline{Q} = \{\theta : I \to I \mid (h'.i)\theta = h'.(i)\theta = (i)\theta h'$ for all Q homomorphism h' in $H'\}$

We claim that $\overline{Q}$ is a left near ring of right quotients of N. For this take $0 \neq q_1$ in $\overline{Q}$ and q_2 any element in $\overline{Q}$. As in Lemma IV.1.7, it can be shown that $q_2^{-1}N$ is strictly dense.

We now want to show that $q_1(q_2^{-1}N) \neq 0$. Now $q_1 \neq 0$ implies that there exists i in I such that $(i)q_1 \neq 0$, that is $((1)h)q_1 \neq 0$, for some h in $\mathrm{Hom}_N(I,I)$. Hence $((1)q_1)h \neq 0$, which implies $(1)q_1 \neq 0$. Hence if $(1)q_1 = (1)h'$ for some $h' \in \mathrm{Hom}_Q(I,I)$ we have $(1)h' \neq 0$.

Since $q_2^{-1}N$ is strictly dense, $(1)h'(q_2^{-1}N) \neq 0$ which implies that there exists r in $q_2^{-1}N$ such that $(1)h'r = (1)q_1 r \neq 0$. Hence $q_1.r \neq 0$ which implies $q_1(q_2^{-1}N) \neq 0$. Hence $\overline{Q}$ is a near ring of right quotients of N.

Since $N \subseteq Q \subseteq \overline{Q}$, by the Remark given after IV.1.10 $\overline{Q}$ is a near ring of right quotients of Q.

We now show that $\overline{Q} = Q$. For this let q' be any element of $\overline{Q}$ and h be a N-homomorphism in H, then since $\mathrm{Hom}_Q(I,I) = \mathrm{Hom}_N(I,I)$, h belongs to H'. Hence $(h.i)q' = h(i)q' = ((i)q')h$, which implies q' is in Q. Hence $Q = \overline{Q}$.

We call Q the complete near ring of right quotients of N.

§ 2. Regular near ring of right quotients.

2.1. Proposition : $I_N \cong Q_N$ canonically implies that there exists an anti near ring homomorphism from H to Q.

Proof : Consider $H \overset{\phi}{\to} I \overset{\theta}{\to} Q$ given by $(h)\phi = (1)h = (1)q$ and $((1)q)\theta = q$, then $\phi . \theta$ is well defined and additive. Now $(h_1h_2)\phi \circ \theta$ $= ((1)h_1h_2)\theta = ((1)q)\theta = q$ where $(1)h_1h_2 = (1)q$, and $\{(h_2)\phi\circ\theta\}\{(h_1)\phi\circ\theta\}$ $= ((1)h_2)\theta\ ((1)h_1)\theta = ((1)q_2)\theta\ ((1)q_1)\theta = q_2q_1$, where $(1)h_2 = (1)q_2$, $(1)h_1 = (1)q_1$. We claim that $q = q_2q_1$. For all i in I $(i)q = ((1)h)q$ $= ((1)q)h = ((1)h_1h_2)h$, and $(i)q_2q_1 = ((1)h)q_2q_1 = (((1)q_2)h)q_1 =$ $(1)h_2hq_1 = ((1)q_1)h_2)h = ((1)h_1)h_2h = ((1)h_1h_2)h$. Hence $\phi \circ \theta$ is an anti near ring homomorphism.

2.2. Definition : An N-subgroup $U_N \subseteq N_N$ is called I-N ideal if U_N is normal in I_N.

Remark : I-N ideals are precisely those ideals of N, which are normal in I.

2.3. Definition : A right N-subgroup K of N_N is weakly large if it has non zero intersection with every nonzero I-N ideal.

2.4. Lemma : Let B_N be an N-subgroup of N_N, C be an I-N ideal maximal with respect to $B \cap C = 0$, then C+B is weakly large N subgroup.

Proof : Let D be an I-N ideal such that $(C + B) \cap D = 0$. Now $x \in (C + D) \cap B$ implies $x = c+d \in B$, that is $d = -c + x \in (C+B) \cap D = 0$. So that $d = 0$, hence $c = x \in B \cap C = 0$, that is $x = 0$. Hence $(C+D) \cap B = 0$. So that $C + D \subseteq C$ by the maximality of C (since C+D is an I-N ideal). Hence $D \subseteq C \subseteq C + B$, implies $D \subseteq (C + B) \cap D = 0$, so that $D = 0$. Hence C+B is weakly large.

2.5. Define $J(I_N) = \{ \sum_{j=1}^{n} (-i_j+(1)h_j s_j+ i_j)$ there exist weakly large N-subgroups $K_j \ni (K_j)h_j=0$, $s_j \in$ generating set of N, $i_j \in I$, $h_j \in H \}$

and $J(N_N) = \{ \sum_{j=1}^{n} -r_j+x_j s_j+r_j$: there exist weakly large N-subgroups $K_j \ni x_j K_j= 0$, $s_j \in$ generating set of N, $r_j \in N$, $x_j \in N. \}$

Since $x_j \in N \subseteq I$, we have $x_j =(1)h_j$ for some homomorphism h_j in H, so $J(N_N) \subseteq J(I_N) \cap N_N$. Now suppose r in N belongs to $J(I_N)$, then $r = -0 +(1)h + 0 \in N \cap J(I_N)$ implies there exists a weakly large N-subgroup K such that $(K)h = 0$. Hence $(1)h K = rK = 0$, implies $r \in J(N_N)$, so that

2.6. $J(N_N) = J(I_N) \cap N_N$.

Since $J(I_N)$ is a submodule of I_N and I_N is weakly essential extension of N_N we have that $J(N_N) = 0$ iff $J(I_N) = 0$.

As in the ring theory we say that a near ring N is regular if for every element a there exists an element a' in N such that $a\,a'\,a = a$. ([4], [14]).

We wish to show that if $J(N_N) = 0$ then Q is regular. For this we prove the following :

2.7. Proposition : $J(N_N) = 0$ implies $I = 1.Q$

Proof : Let $i \in I$ be arbitrary and h_1, h_2 be right N homomorphisms in H such that $(r)h_1 = (r)h_2$ for all r in N. Let $i = (1)h'$ where h' is a homomorphism in H.

Consider $B = \{\sum_{j=1}^{n} -i_j + x_j + i_j \mid x_j \in N,\ i_j \in I\}$ and define

$$h'' : B \to I \text{ by } \Big(\sum_{j=1}^{n} -i_j + x_j + i_j\Big)h'' = \sum_{j=1}^{n} -i_j + (x_j)h' + i_j .$$

We note that h'' is additive and that $h''|_N = h'|_N$. Now

$b = \sum_{j=1}^{n} -i_j + x_j + i_j = 0$ implies $(-i_1 + x_1 + i_1) + (-i_2 + x_2 + i_2) + \ldots +$

$$+ (-i_{n-1} + x_{n-1} + i_{n-1}) = -i_n - x_n + i_n ,$$

Or $\{(i_n - i_1) + x_1 + (i_1 - i_n)\} + \ldots + \{(i_n - i_{n-1}) + x_{n-1} + (i_{n-1} - i_n)\} = -x_n \in N$.

So that $(\text{L.H.S.})h' = (-x_n)h'$, or $(\text{L.H.S.})h'' = -(x_n)h'$.

or $\{(i_n - i_1) + (x_1)h' + (i_1 - i_n)\} + \ldots + \{(i_n - i_{n-1}) + (x_{n-1})h' + (i_{n-1} - i_n)\} = -(x_n)h'$

or $\{-i_1 + (x_1)h' + i_1\} + \ldots + \{-i_{n-1} + (x_{n-1})h' + i_{n-1}\} = -i_n - (x_n)h' + i_n$.

So that $(b)h'' = 0$, hence h'' is well defined.

Since h'' is additive and $(bs)h'' = (b)h''s$ for all distributive s in N, it follows that h'' is a right N-homomorphism. Let $A = \{\sum_{j=1}^{n} (-i_j + x_j + i_j) \in B \mid \sum_{j=1}^{n} (-i_j + (x_j)h' + i_j) \in N\}$, then $A \subseteq B$.

Case I: Suppose $A \cap N$ is weakly large, then since $(x)h'' \in N$ for all $x \in A \cap N$ we have $(x)h'' \circ (h_1 - h_2) = (x)h''h_1 - (x)h''h_2 = 0$.

Hence by the definition of $J(I_N)$ we have $(1)h'' \circ (h_1 - h_2) \in J(I_N) = 0$. Hence $(1)h' \circ (h_1 - h_2) = 0$, implies $(1)h'h_1 = (1)h'h_2$, so that $(i)h_1 = (i)h_2$, that is $i \in 1.Q$. Hence $I = 1.Q$.

Case II: Suppose $A \cap N$ is not weakly large, this implies that there exists a non-zero I-N ideal U maximal with respect to $(A \cap N) \cap U = 0$. So that by Lemma IV.2.4 $U + (A \cap N)$ is weakly large. Now $D_N = \{\sum_{j=1}^{n} (-i_j + (u_j)h' + i_j) \mid u_j \in U, i_j \in I\}$ is a submodule of I_N. Now x is in $D \cap N$ implies that $x = \sum_{j=1}^{n} (-i_j + (u_j)h' + i_j)$ is in N for some $u_j \in U$, $i_j \in I$ $(1 \leq j \leq n)$. Then $b = \sum_{j=1}^{n} (-i_j + u_j + i_j) \in U \cap A$ implies $b = 0$ (since $0 = (A \cap N) \cap U = A \cap U$). So that $(b)h'' = 0 =$ $\sum_{j=1}^{n} (-i_j + (u_j)h' + i_j) = x$. Hence $D_N \cap N_N = 0$, which implies $D_N = 0$ since I_N is a weakly essential extension of N_N. Thus $(U)h' = 0$ since $(U)h' \subseteq D_N$.

Now for all $u+x \in U + (A \cap N)$ we have $(u+x)h'' \circ (h_1 - h_2) = (x)h'' \circ (h_1 - h_2) = (x)h''h_1 - (x)h''h_2 = 0$, since $(x)h''$ is in N. So that $(1)h'' \circ (h_1 - h_2) \in J(I_N) = 0$, that is $(i)h_1 = (i)h_2$. Hence $i \in 1.Q$, so that $I = 1.Q$.

2.8. Corollary : Let N be a near ring such that $J(N_N) = 0$. If ϕ and θ are given as in Proposition IV. 2.1 then $\phi \circ \theta$ is an anti near ring homomorphism.

2.9. Proposition : $J(N_N) = 0$ implies that given a homomorphism h in H, there exists h' in H such that $h\, h'\, h = h$.

Proof : Let $h^{-1}(0) = \{x \in N \mid (x)h = 0\}$. For all homomorphisms h' in H, $(x)(h\, h'h - h) = 0$ for all x in $h^{-1}(0)$. Hence if $h^{-1}(0)$ is weakly large then $(1)(h\, h'h - h) \in J(I_N) = 0$, that is $(1)h\, h'h = (1)h$. Hence $(r)h\, h'h = (r)h$ for all r in N. Thus we get $(i)\, h\, h'\, h = (i)h$ for all i in I (since $I = 1.Q$) which implies that $h\, h'h = h$.

If $h^{-1}(0)$ is not weakly large then there exists a non-zero I-N ideal K maximal with respect to $K \cap h^{-1}(0) = 0$, such that $K + h^{-1}(0)$ is weakly large. Define $\psi : (K)h \to I$ by $((k)h)\psi = k$. It is easy to see that ψ is well defined. Let $B = \{ \sum_{j=1}^{n} (-i_j + (k_j)h + i_j) \mid i_j \in I, k_j \in K\}$. Then B is an N-submodule of I. Define $\bar{\psi} : B \to I$ by

$$\Big(\sum_{j=1}^{n} (-i_j + (k_j)h + i_j)\Big)\bar{\psi} = \sum_{j=1}^{n} (k_j)h\psi = \sum_{j=1}^{n} k_j \, .$$

Note that $\bar{\psi}$ is additive and $\bar{\psi}|_{(k)h} = \psi$. Making use of these observations it is a

routine matter to check that $\bar{\psi}$ is a well defined right N-homomorphism. By injectivity of I_N there exists a N-homomorphism $h' : I_N \to I_N$ such that $h'|_B = \bar{\psi}$, which implies that $h'|_{(K)h} = \bar{\psi}|_{(K)h} = \psi|_{(K)h}$ and so $h\,h'$ is identity on K.

Now for all $k+x \in K + h^{-1}(0)$, $(k+x)\,(h\,h'h - h) = (k+x)h\,h'h - (k+x)h = (k)h\,h'h - (k)h = (k)h - (k)h = 0$ implies that $(1)\,(h\,h'h - h) \in J(I_N) = 0$, that is $(1)h\,h'h = (1)h$. So that $(r)h\,h'h = (r)h$ for all r in N, which implies $(i)h\,h'h = (i)h$ for all $i \in I$ (since $I = 1.Q$). Hence $h\,h'h = h$.

2.10. Theorem : If $J(N_N) = 0$ then Q is a regular near ring.

Proof : Now q in Q implies $(1)q = (1)h$ for some N-homomorphism h in H. In view of IV. 2.9 there exists h' in H such that $h\,h'h = h$. Let $(h)\phi \circ \theta = q$, then $q = (h)\phi \circ \theta = (h\,h'h)\phi \circ \theta$ where ϕ and θ are as defined in Proposition IV. 2.1. Since $\phi \circ \theta$ is an anti near-ring homomorphism we get

$$q = \{(h'h)\,\phi \circ \theta\}\ \{(h)\,\phi \circ \theta\}$$

$$= \{(h)\,\phi \circ \theta\}\{(h')\,\phi \circ \theta\}\ \{(h)\,\phi \circ \theta\}$$

$$= q\,q'q \quad \text{where} \quad (1)q' = (1)h'.$$

Hence Q is regular.

§ 3. Relationship between classical near ring of right quotients and a complete near ring of right quotients.

In this section we give a sufficient condition for a complete near ring of right quotients to be a classical near ring of right quotients.

Let N be a left d.g. near ring generated by the semi group of all the distributive elements of N, and S' be a semi-group of some distributive non-zero divisors of N. Suppose that N satisfies the following conditions with respect to S'

(i) C.R.M.P.: Given a in N, λ in S' there exist a' in N and λ' in S' such that $a\lambda' = \lambda a'$.

(ii) If in (i) a, λ both belong to S' then λ', a' both are distributive.

Suppose that every weakly large right N-subgroup of N_N contains an element of S'. We claim that Q is a classical near ring of right quotients of N with respect to S'. For this we first prove the following:

3.1 Lemma : $\lambda \in S' \implies \lambda N$ is a dense right N-subgroup.

Proof : Let $r_1 \neq 0$ and r_2 belong to N. Given $\lambda \in S'$, $r_2 \in N$ there exist λ' in S' and r in N such that $\lambda r = r_2\lambda' \in \lambda N$. That is there exists λ' in N such that $r_1\lambda' \neq 0$, $r_2\lambda' \in \lambda N$.

3.2. Lemma : A right N-subgroup D_N of N_N is dense implies that it is weakly large.

Proof : $U \neq 0$ I-N ideal implies there exists $0 \neq u \varepsilon U \subseteq N$, so that there exists r in N such that $0 \neq ur \varepsilon D_N$, so that $0 \neq ur \varepsilon U \cap D$. Hence D_N is weakly large.

3.3. Lemma : $J(N_N) = 0$.

Proof : Now $x \varepsilon J(N_N)$ implies that $x = \sum_{j=1}^{n} (-r_j+x_j s_j+ r_j)$ for some $r_j \varepsilon N$, $x_j \varepsilon N$ and s_j is a distributive element of N $(1 \leq j \leq n)$ and there exist weakly large right N-subgroups K_j with $x_j K_j = 0$. Since each K_j contains an element of S' say λ_j then $x_j \lambda_j = 0$ implies that $x_j = 0$ for all $j = 1,2,\ldots n$. So that $x = 0$. Hence $J(N_N) = 0$.

3.4. Corollary : Q is a regular near ring.

3.5. Lemma : λ in S' implies that λ is a distributive non-zero divisor in Q.

Proof : Now $\lambda q = 0$ implies $\lambda q(q^{-1}N) = 0$, which implies $q(q^{-1}N)= 0$. So that $qr = 0$ for all r in $q^{-1}N$ which is strictly dense. Therefore, $(1)q.r = 0$, that is $(1)qr = (1)hr = 0$ for all r in $q^{-1}N$ and for some N-homomorphism h in H. Hence $(1)h = 0$, that is $q = (1)q = 0$.

Also $q\lambda = 0$ implies that $q\lambda N = 0$, that is $(1)q\lambda r = 0$ for all r in N. So $(1)h\lambda r = 0$ for all r in N and for some N-homomorphism h in H. Therefore $(\lambda N)h = 0$. Since λN is dense and hence weakly large, we have that $(1)h \varepsilon J(I_N) = 0$. Hence $q = (1)q = 0$.

We now show that λ is distributive in Q. For this let q_1, q_2 be any two elements in Q and i be any arbitrary element of I, then

$(i)(q_1+q_2)\cdot\lambda = ((i)q_1+(i)q_2)\lambda = (i)q_1\lambda + (i)q_2\lambda = (i)(q_1\lambda + q_2\lambda)$. So that λ is distributive in Q.

3.6. Lemma : Every distributive non zero divisor of Q is invertible in Q.

Proof : Let q be a distributive non zero divisor of Q. Since Q is regular, there exists q' in Q such that q q'q = q, which implies that q(q'q-1) = 0, that is q'q = 1.

Also (q q'-1)q = 0 implies that q q'=1. Hence q is invertible in Q.

3.7. Theorem : Q is a classical near ring of right quotients of N with respect to S'.

Proof : By Lemmas IV.3.5, IV.3.6 we have that every element of S' is invertible in Q. It remains to show that q in Q implies $q = a\lambda^{-1}$ for some a in N and some λ in S'.

Now q in Q implies that $q^{-1}N$ is strictly dense, hence dense and hence weakly large. So $q^{-1}N$ contains an element of S', say λ. Thus $q\lambda = a$ is in N, that is $q = a\lambda^{-1}$. Hence Q is a classical near ring of right quotients of N.

Now suppose that instead of condition (ii), N satisfies the condition

(ii)' For every ν in S' and every r in N, νr distributive implies r is distributive.

The other conditions on N remaining same as given in the beginning of this section.

We then prove

3.8. Proposition : Suppose every N-homomorphism $f: U \to N$, where U is a right ideal of N, can be extended to an N-homomorphism $g:J \to N$, where J is a right ideal of N containing U and containing an element of S', then the complete near ring of right quotients of N coincides with classical near ring Q' of right quotients of III.1.5.

Proof : In this case by 3.10 Q'_N is injective, also $N_N \leq Q'_N$. So we have that I_N is contained in Q'_N .

We now show that Q'_N is a weakly essential extension of N_N. For this let $0 \neq U$ be a submodule of Q'_N . Then there exists $0 \neq u \in U \subseteq Q'_N$, which implies that $0 \neq u = a\lambda^{-1}$ where a is in N and λ is in S', so that $0 \neq u\lambda = a \in N \cap U$. Hence Q'_N is a weakly essential extension of N_N. So that $Q'_N = I_N$.

Let $q' = a\lambda^{-1}$ be any element of Q'. Suppose h_1, h_2 are right N-homomorphisms in H such that $(r)h_1 = (r)h_2$ for all r in N. Then $(q'\lambda)h_1 = (q'\lambda)h_2 \in I = Q'$ implies $(q'\lambda)h_1\lambda^{-1} = (q'\lambda)h_2\lambda^{-1}$, so that $(q')h_1 = (q')h_2$. This implies that $q' \in 1.Q$, that is $I_N = Q'_N \subseteq 1.Q \subseteq I_N$. Hence $I = Q' = 1.Q$.

Now $\phi : Q \to I = Q'$ given by $(q)\phi = (1)q$ is a monomorphism and since $I = 1.Q$ we have that it is also epimorphism.

It remains to show that ϕ is a near ring homomorphism. We already know that ϕ is additive. To prove that ϕ preserves multiplication, we first note that $(qr)\phi = (q.\phi_r)\phi = (1)q \circ \phi_r = (1)qr = (q)\phi(1)\phi_r = (q)\phi\ (\phi_r)\phi = (q)\phi\ (r)\phi$. Now for any q_1, q_2 in Q we have $(q_1q_2r)\phi = (q_1)\phi\ (q_2r)\phi$ for all r in $q_2^{-1}N$. This implies that $(q_1q_2)\phi\ r = (q_1)\phi(q_2)\phi r$. Thus there exist h, h_1, h_2, h', right N-homomorphisms in H such that $(q_1q_2)\phi = (1)h, (q_1)\phi = (1)h_1, (q_2)\phi = (1)h_2, (q_1)\phi\ (q_2)\phi = (1)h'$ and $(1)hr = (1)h_1(1)h_2r = (1)h'r$ for all r in $q_2^{-1}N$. Since $q_2^{-1}N$ is strictly dense, this gives that $(1)h = (1)h'$. So $(q_1q_2)\phi = (q_1)\phi(q_2)\phi$. Thus $Q \cong Q'$ as near rings.

§ 4. Complete near ring of left quotients

Let N be a left near ring with unity.

4.1. A subset D of N is called a left N-set if $ND = \{rd \mid r \in N, d \in D\} \subseteq D$.

4.2. A left N-set D is dense if for all $0 \neq r_1$ in N and all r_2 in N there exists r in N such that $rr_1 \neq 0$ and $rr_2 \in D$.

Remark : We note that

(1) Intersection of dense N-sets is again a dense N-set.

(2) Any N-set containing a dense N-set is dense.

To construct a left near ring of left quotients of N, we proceed as follows:

Let $X = \bigcup \{\theta : D \to N \mid (rd)\theta = r(d)\theta$ for all r in N and all d in D where D is a dense left N-subgroup$\}$

If $\theta_i : D_i \to N$ $(i=1,2)$ are in X, define $\theta_1 \sim \theta_2$ iff there exists a dense N-set A in $D_1 \cap D_2$ such that θ_1 and θ_2 coincide on A. It is easy to see that '$\sim$' is an equivalence relation.

Let C denote the set of equivalence classes of X with respect to the equivalence relation defined above. Let $[\theta_i]$ $(i=1,2)$ be in C with $\theta_i : D_i \to N$. Define $[\theta_1] + [\theta_2] = [\theta_1 + \theta_2]$ where $\theta_1 + \theta_2 : D_1 \cap D_2 \to N$ is such that $(x)(\theta_1+\theta_2) = (x)\theta_1 + (x)\theta_2$ for all x in $D_1 \cap D_2$. It can be easily seen that addition is well defined and that C is an additive group.

To define multiplication in C, take $[\theta_i]$ $(i=1,2)$ in C where θ_i has domain D_i. Let $A = \{x \in D_1 \mid (x)\theta_1 \in D_2\}$ then $NA \subseteq A$ and A is dense; for if $r_1 \neq 0$, r_2 are in N, then there exists r in N such that $rr_1 \neq 0$, $rr_2 \in D_1$. Given $rr_1 \neq 0$ and $(rr_2)\theta_1$ in N, there exists r' in N such that $r'r\,r_1 \neq 0$ and $r'(rr_2)\theta_1 = (r'r\,r_2)\theta_1 \in D_2$. This implies that $r'r\,r_1 \neq 0$ and $r'rr_2 \in A$.

Consider $\overline{A} = \{\sum_{i=1}^{n} \pm a_i \mid a_i \in A\}$. Then $\overline{A}$ is a dense left N-subgroup. Define $\theta_1' : \overline{A} \to N$ by $(\sum_{i=1}^{n} \pm a_i)\,\theta_1' = \sum_{i=1}^{n} \pm (a_i)\theta_1$. Observing that θ_1' is additive and that it coincides with θ_1 on A, it can be proved that θ_1' is a well defined left N-homomorphism.

Define $[\theta_1][\theta_2] = [\theta_1' \circ \theta_2]$. To check that this multiplication is well defined, suppose that $\theta_i \sim \phi_i$ $(i=1,2)$ where the domain of ϕ_i is B_i. Now $\theta_i \sim \phi_i$ implies there exists a dense left

N-set $A_i \subseteq D_i \cap B_i$ on which θ_i and ϕ_i coincide ($i=1,2$). If $U = \{x \in A_1 \mid (x)\theta_1 = (x)\phi_1 \in A_2\}$, then U is a dense left N-set contained in (domain $\theta_1' \circ \theta_2$) $\cap$ (domain $\phi_1' \circ \phi_2$). Moreover $\theta_1' \circ \theta_2$ coincides with $\phi_1' \circ \phi_2$ on U. Hence the multiplication is well defined.

It is a routine matter to check that C, together with the above defined addition and multiplication, is a left near ring. The mapping $N \to C$ given by $n \to [\phi_n]$ is a monomorphism, where $\phi_n : N \to N$ is given by $(x)\phi_n = xn$. Hence N may be considered as contained in C.

We claim that ${}_NC$ is a weakly essential extension of ${}_NN$. For this we first prove the following:

4.3. Lemma : For all $0 \neq c_1$ in C and all c_2 in C there exists r in N such that $rc_1 \neq 0$ and rc_2 is in N.

Proof : Suppose $0 \neq c_1 = [\theta_1]$, $c_2 = [\theta_2]$ with $\theta_1 : D_1 \to N$ and $\theta_2 : D_2 \to N$. Now $[\theta_1] \neq 0$ implies, for all dense left N-sets $A \subseteq D_1 \cap D_2$, $(A)\theta_1 \neq 0$. Given $(a)\theta_1 \neq 0$, $a \in A \subseteq N$, there exists r' in N such that $r'(a)\theta_1 = (r'a)\theta_1 \neq 0$ and $r'a \in D_1 \cap D_2$. Now $r'a\, c_2 = [\phi_{r'a}]\, [\theta_2] =$ $[\phi'_{r'a} \circ \theta_2] = [\phi_{(r'a)\theta_2}] \in N$. And $r'ac_1 = 0$ implies $[\phi_{r'a}][\theta_1] = [\phi'_{r'a} \circ \theta_1] = 0$. This implies there exists a dense left N-set A' such that $(A')\phi'_{r'a} \circ \theta_1$ $= (A'r'a)\theta_1 = 0$. So $A'(r'a)\theta_1 = 0$, that is $(r'a)\theta_1 = 0$, which is a contradiction. Hence $r'ac_1 \neq 0$. Put $r'a = r$ and we have the result.

4.4. Corollary : ${}_N C$ is a weakly essential extension of ${}_N N$.

Let ${}_N I$ denote the maximal weakly essential extension of ${}_N N$.

4.5. Definition : A left N-set D is strictly dense iff for any pair of left N-homomorphisms $h_1, h_2 : {}_N I \to {}_N I$, $(d)h_1 = (d)h_2$ for all d in D implies $(r)h_1 = (r)h_2$ for all r in N.

4.6. Proposition : A left N-set D is dense implies that D is strictly dense.

Proof : Let $h_1, h_2 \in \mathrm{Hom}_N({}_N I, {}_N I)$ such that $(d)h_1 = (d)h_2$ for all d in D. Let A denote the left N-submodule of ${}_N I$ generated by $(N)(h_1 - h_2)$. Now $0 \neq r_1 \in A \cap N$ implies $0 \neq r_1 = \sum_{j=1}^{n} (-i_j + a_j + i_j) \in N$ where $i_j \in I$ and $a_j \in (N)(h_1 - h_2) \cup \{-(N)(h_1 - h_2)\}$. Thus

$$0 \neq r_1 = \{-i_1 + (x_1)(h_1 - h_2) + i_1\} + \{-i_2 - (x_2)(h_1 - h_2) + i_2\} + \dots + \{-i_n + (x_n)(h_1 - h_2) + i_n\}$$

Given $r_1 \neq 0$, x_1 in N there exists b_1 in N such that $b_1 r_1 \neq 0$, $b_1 x_1 \in D$, so that we have

$$0 \neq b_1 r_1 = \{-b_1 i_1 + (b_1 x_1)(h_1 - h_2) + b_1 i_1\} + \{-b_1 i_2 - (b_1 x_2)(h_1 - h_2) + b_1 i_2\} + \dots + \{-b_1 i_n + (b_1 x_n)(h_1 - h_2) + b_1 i_n\}$$

$$= \{-b_1 i_2 - (b_1 x_2)(h_1 - h_2) + b_1 i_2\} + \dots + \dots + \{-b_1 i_n + (b_1 x_n)(h_1 - h_2) + b_1 i_n\}$$

Now taking $0 \neq b_1 r_1$, $b_1 x_2$ there exists b_2 in N such that $b_2 b_1 r_1 \neq 0$, $b_2 b_1 x_2 \in D$. Hence

$$0 \neq b_2 b_1 r_1 = \{-b_2 b_1 i_3 + (b_2 b_1 x_3)(h_1 - h_2) + b_2 b_1 i_3\} + \dots +$$
$$+ \{-b_2 b_1 i_n + (b_2 b_1 x_n)(h_1 - h_2) + b_2 b_1 i_n\}$$

Proceeding in this way we get $0 \neq b_n b_{n-1} \dots b_2 b_1 r_1 = 0$, a contradiction. Hence $A \cap N = 0$ which implies that $A = 0$, so that $(N)(h_1 - h_2) = 0$, that is $(n)h_1 = (n)h_2$ for all n in N. Hence D is strictly dense.

4.7. Definition : A left near ring $S \supseteq N$ is a left near ring of left quotients of N iff for all $0 \neq s_1$ in S and all s_2 in S, $(s_2^{-1}N)s_1 \neq 0$.

Remark 1: The above definition implies that for all s in S, $s^{-1}N$ is dense and hence strictly dense. Hence we do not require the condition, that for all s in S, $s^{-1}N$ is strictly dense, in the definition of left near ring of left quotients.

Remark 2 : Any classical left near ring of left quotients of N is a left near ring of left quotients of N.

It is an immediateconsequence of Lemma 4.3 that C is a left near ring of left quotients of N.

4.8. Proposition : Any left near ring of left quotients of N is contained in C.

Proof : Let S be any left near ring of left quotients of N. For any s in S let $s^{-1}N = \{r \in N \mid rs \in N\}$. Then $s^{-1}N$ is dense, since for any $r_1 \neq 0$

in N and r_2 in N, considering $r_1 \neq 0 \in N \subseteq S$, $r_2 s \in S$ there exists $r \in (r_2 s)^{-1}N$ such that $rr_1 \neq 0$, which implies $rr_2 s \in N$, $rr_1 \neq 0$. Hence there exists $r \in N$ such that $rr_1 \neq 0$, $rr_2 \in s^{-1}N$. Define $\phi_s : s^{-1}N \to N$ by $(x)\phi_s = xs$ for all x in $s^{-1}N$ and $\phi'_s : \{\sum_{i=1}^{n} \pm r_i \mid r_i \in s^{-1}N\} \to N$ by $(\sum_{i=1}^{n} \pm r_i)\phi'_s = \sum_{i=1}^{n} \pm (r_i)\phi_s$. Observing that ϕ'_s is additive and that it coincides with ϕ_s on $s^{-1}N$, it is easy to see that ϕ'_s is well defined. Since $s^{-1}N$ is dense, the domain of ϕ'_s is also dense and hence $[\phi'_s]$ belongs to C. Now define $\phi : S \to C$ by $(s)\phi = [\phi'_s]$. Let $s_1 = s_2$ in S, then $(x)\phi'_{s_1} = (x)\phi'_{s_2}$ for all x in $s_1^{-1}N \cap s_2^{-1}N$ implies ϕ is well defined. For all $x \in s_1^{-1}N \cap s_2^{-1}N$, $(x)\phi'_{s_1+s_2} = (x)(\phi'_{s_1} + \phi'_{s_2})$ implies ϕ is additive, also since $(x)\phi'_{s_1 s_2} = (x)\phi'_{s_1} \circ \phi'_{s_2}$ for all $x \in (s_1 s_2)^{-1}N \cap s_1^{-1}N$, ϕ is a near ring homomorphism . Now $(s)\phi = 0$ implies $[\phi'_s] = 0$, that is there exists a dense left N-set $A \subseteq s^{-1}N$ such that $(A)\phi_s = 0$. That is $As = 0$, implies $s = 0$. Hence ϕ is a monomorphism.

4.9. Proposition : If S is a left near ring of left quotients of N and T is a left near ring of left quotients of S, then T is a near ring of left quotients of N.

Proof : Let $t_1 \neq 0$, t_2 be elements of T, then there exists $s \in S$ such that $0 \neq st_1 \in S$. Considering $0 \neq s\,t_1$, st_2 in T we have s' in S such that $0 \neq s's\,t_1$ and $s's\,t_2 \in S$. Now $0 \neq s's\,t_1$, s's in S implies there exists r in N such that $r\,s's\,t_1 \neq 0$, $r\,s's \in N$.

Again $r\ s's\ t_1 \neq 0$, $r\ s's\ t_2$ in S implies there exists $r' \varepsilon\ N$ such that $r'r\ s's\ t_1 \neq 0$, $r'r\ s's\ t_2\ \varepsilon\ N$. Hence the result.

If we go through the same procedure again, starting with C in place of N and obtain the left near ring C_o , then we have the following:

4.10. Proposition : $C = C_o$

Proof : Clearly $C \subseteq C_o$. Making use of the Proposition 4.9 we get that C_o is a near ring of left quotients of N and by Proposition 4.8 $C_o \subseteq C$. Hence $C = C_o$.

C is called the complete near ring of left quotients of N.

CHAPTER V

GENERALISED CENTRALIZERS OF NEAR-RING MODULES.

In this chapter we have constructed generalized centralizer of a near-ring module. For this we have taken a collection τ' of N-subgroups of a module M(over a d.g. near ring N) satisfying the conditions:

(a) If $S \in \tau'$ and $S \subseteq T$ then $T \in \tau'$

(b) If S and $T \in \tau'$ then $S \cap T \in \tau'$

(c) If $S, T \in \tau'$ and $\alpha : S \to M$ is such that $(sr)\alpha = (s)\alpha r$ for all $s \in S$ and all distributive $r \in N$ and

$$B = \{ \sum_{i=1}^{n} \pm x_i | x_i \in S \text{ and } \sum_{i=1}^{n} \pm (x_i)\alpha \in T\} \text{ then } B \in \tau',$$

and with each such τ' we have associated a near ring P'. This near ring turns out to be non-associative. If τ' is a collection of large N-subgroups then τ' satisfies conditions (a), (b) and (c) and thus we can associate with this collection a non-associative near ring.

In case N is a ring and τ' is the collection of large submodules of a given module, then the near ring P' turns out to be an (associative) ring and this ring is actually Johnson's extended centralizer of a module. ([11]).

Above conditions (a), (b) and (c) are dual to the conditions :

(i) $S \in \tau$, $T \subseteq S$ then $T \in \tau$

(ii) $S, T \in \tau$ implies $S + T \in \tau$

(iii) Let $S, T \in \tau$ and $\alpha : M \to M/T$ be a mapping such that $\alpha(mr) = \alpha(m)r$ for all m in M and all distributive r in N. Let U/T be the submodule of M/T generated by

$\alpha(S) \cup \{\alpha(m+m') - \alpha(m') - \alpha(m)\}$, then $U \in \tau$

These conditions are always satisfied by small submodules. One can associate a near ring say P with this collection also.

It is proved here that if the near-ring module M is τ'-complemented (τ - complemented) in a certain sense then the near ring $P'(P)$ is regular.

§ 1. Generalised Centralizers of Near-Ring Modules.

We assume that τ is a non-empty collection of proper N-submodules of M satisfying (i), (ii) and (iii).

$$\text{Let } P_T = \{\alpha : M \to \frac{M}{T} \mid \alpha(mr) = \alpha(m)r \text{ for all } m \text{ in } M \text{ and all distributive } r \text{ in } N\}$$

and $X = \bigcup_{T \in \tau} P_T$. If $\beta \in P_T$ and $\beta' \in P_{T'}$ let

$\frac{U}{T}$ = submodule of $\frac{M}{T}$ generated by $\{\beta(m+m') - \beta(m') - \beta(m)\}$.

$\frac{U'}{T'}$ = submodule of $\frac{M}{T'}$ generated by $\{\beta'(m+m') - \beta'(m') - \beta'(m)\}$.

Define $\beta \sim \beta'$ iff there exists $T_1 \in \tau$ such that $T_1 \supseteq U+U' \supseteq T+T'$ and $\pi_{T_1T}\, \beta = \pi_{T_1T'}\, \beta'$ where $\pi_{T_1T} : \frac{M}{T} \to \frac{M}{T_1}$ is the natural homomorphism given by $\pi_{T_1T}(m+T) = m+T_1$.

It is easily seen that this relation is reflexive and symmetric. To see that it is also transitive, consider $\beta \in P_T, \beta' \in P_{T'}$, $\beta'' \in P_{T''}$ such that $\beta \sim \beta'$ and $\beta' \sim \beta''$. This implies that there exists $T_1 \in \tau$ such that $T_1 \supseteq U+U' \supseteq T+T'$ and $\pi_{T_1T}\beta = \pi_{T_1T'}\beta'$. Also there exists $T_2 \in \tau$ with $T_2 \supseteq U'+U'' \supseteq T'+T''$ such that $\pi_{T_2T'}\beta'$ $= \pi_{T_2T''}\beta''$ where U''/T'' = submodule of M/T'' generated by $\{\beta''(m+m') - \beta''(m') - \beta''(m)\}$.

Now $T_3 = T_1 + T_2 \supseteq U + U' + U'' \supseteq U + U'' \supseteq T + T''$ and $T_3 \in \tau$. Moreover $\pi_{T_3T_1}\pi_{T_1T}\beta = \pi_{T_3T_1}\pi_{T_1T'}\beta'$, this implies $\pi_{T_3T}\beta = \pi_{T_3T'}\beta'$.

And $\pi_{T_3T_2}\pi_{T_2T'}\beta' = \pi_{T_3T_2}\pi_{T_2T''}\beta''$, which implies $\pi_{T_3T'}\beta' = \pi_{T_3T''}\beta''$. Hence $\pi_{T_3T}\beta = \pi_{T_3T''}\beta''$ which implies that the above defined relation is transitive and thus an equivalence relation.

Let P denote the set of equivalence classes of X. P can be made into a near ring as follows:

To define addition in P take $\alpha \in P_S$ and $\beta \in P_T$. Put $U = S+T$. Then define $[\alpha] + [\beta] = [\pi_{US}\alpha + \pi_{UT}\beta] = [\gamma]$ say where $\gamma = \pi_{US}\alpha + \pi_{UT}\beta$

To see that this addition is well defined consider $\alpha' \in P_{S'}$ and $\beta' \in P_{T'}$ such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Then if $U' = S' + T'$ we have $[\alpha'] + [\beta'] = [\pi_{U'S'}\alpha' + \pi_{U'T'}\beta'] = [\delta]$ where $\delta = \pi_{U'S'}\alpha' + \pi_{U'T'}\beta'$.

Now let, $\frac{V}{S}$ = submodule of $\frac{M}{S}$ generated by $\{\alpha(m+m')-\alpha(m')-\alpha(m)\}$

$$\frac{V'}{S'} = \text{submodule of } \frac{M}{S'} \text{ generated by } \{\alpha'(m+m') - \alpha'(m') - \alpha'(m)\}$$

$$\frac{W}{T} = \text{submodule of } \frac{M}{T} \text{ generated by } \{\beta(m+m') - \beta(m') - \beta(m)\}$$

and $$\frac{W'}{T'} = \text{submodule of } \frac{M}{T'} \text{ generated by } \{\beta'(m+m') - \beta'(m') - \beta'(m)\}$$

Now $[\alpha] = [\alpha']$ implies there exists $T_1 \in \tau$ such that $T_1 \supseteq V+V' \supseteq S+S'$ and $\pi_{T_1 S}\alpha = \pi_{T_1 S'}\alpha'$, and $[\beta] = [\beta']$ implies there exists $T_2 \in \tau$ such that $T_2 \supseteq W+W' \supseteq T+T'$ and $\pi_{T_2 T}\beta = \pi_{T_2 T'}\beta'$.

Let $$\frac{Y}{U} = \text{submodule of } \frac{M}{U} \text{ generated by } \{\gamma(m+m') - \gamma(m') - \gamma(m)\}$$

and $$\frac{Y'}{U'} = \text{submodule of } \frac{M}{U'} \text{ generated by } \{\delta(m+m') - \delta(m') - \delta(m)\}.$$

We have to show that there exists $T_3 \in \tau$ such that $T_3 \supseteq Y + Y' \supseteq U+U'$ and such that $\pi_{T_3 U}\gamma = \pi_{T_3 U'}\delta$. Take $T_3 = Y+Y'+T_1+T_2$, then $T_3 \in \tau$ and $T_3 \supseteq Y+Y' \supseteq U+U'$.

Now
$$\begin{aligned}
\pi_{T_3 U}\gamma(m) &= \pi_{T_3 U}(\pi_{US}\alpha(m) + \pi_{UT}\beta(m)) \\
&= \pi_{T_3 U}(\pi_{US}(x+S) + \pi_{UT}(y+T)) \\
&= \pi_{T_3 U}(x+U + y+U) \\
&= (x+y) + T_3 \quad \text{where } \alpha(m) = x+S \text{ and } \beta(m) = y+T.
\end{aligned}$$

And
$$\begin{aligned}
\pi_{T_3 U'}\delta(m) &= \pi_{T_3 U'}(\pi_{U'S'}\alpha'(m) + \pi_{U'T'}\beta'(m)) \\
&= \pi_{T_3 U'}(x'+U'+y'+U') \\
&= (x'+y') + T_3 \text{ where } \alpha'(m) = x'+S' \text{ and } \beta'(m) = y'+T'.
\end{aligned}$$

Also $\pi_{T_1S}\alpha(m) = \pi_{T_1S'}\alpha'(m)$ implies $x+T_1 = x'+T_1$, that is $x-x' \varepsilon T_1 \subseteq T_3$, and $\pi_{T_2T}\beta(m) = \pi_{T_2T'}\beta'(m)$ implies $y+T_2 = y'+T_2$, that is $y-y' \varepsilon T_2 \subseteq T_3$.

But $(x+y) - (x'+y') - (x-x') = x+y-y'-x'+x'-x = x+(y-y')-x \varepsilon T_3$ (since $y-y' \varepsilon T_3$ and T_3 is normal in M) implies that $(x+y) - (x'+y') \varepsilon T_3$ (since $(x-x') \varepsilon T_3$). So $(x+y) + T_3 = (x'+y') + T_3$ and hence addition is well defined.

If 0_S and 0_T denote the zero mappings in P_S and P_T respectively, then it can be easily seen that $[0_S] = [0_T]$ is the zero element of P. Observing that $[-\alpha] = -[\alpha]$ (where $(-\alpha)(x) = -\alpha(x)$) we see that P is an additive group.

Multiplication in P is defined as follows: Suppose $\alpha \varepsilon P_S$ and $\beta \varepsilon P_T$. Let C/T be the submodule of M/T generated by $\beta(S) \cup \{\beta(m+m')-\beta(m')-\beta(m)\}$. Consider $M \xrightarrow{\beta} \frac{M}{T} \xrightarrow{\mu} \frac{M}{T} | \frac{C}{T} \xrightarrow{g} \frac{M}{C}$ where μ and g are the natural homomorphisms. Since $g\mu\beta$ is a homomorphism which vanishes on S, it induces a homomorphism $\tilde{\beta} : \frac{M}{S} \to \frac{M}{C}$. Define $[\beta]\ [\alpha] = [\tilde{\beta}\ \alpha]$. To prove that the multiplication is well defined, consider $\alpha' \varepsilon P_{S'}$ and $\beta' \varepsilon P_{T'}$ such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Let $\frac{C'}{T'}$ be the submodule of $\frac{M}{T'}$ generated by $\beta'(S') \cup \{\beta'(m+m')-\beta'(m')-\beta'(m)\}$, and let $\tilde{\beta}' : \frac{M}{S'} \to \frac{M}{C'}$ be the homomorphism induced by

$M \xrightarrow{\beta'} \frac{M}{T'} \xrightarrow{\mu'} \frac{M}{T'} | \frac{C'}{T'} \xrightarrow{g'} \frac{M}{C'}$. We have to show that $[\tilde{\beta}\alpha] = [\tilde{\beta}'\alpha']$.

Let $\frac{U}{S}$ = submodule of $\frac{M}{S}$ generated by $\{\alpha(m+m') - \alpha(m') - \alpha(m)\}$

$\frac{U'}{S'}$ = submodule of $\frac{M}{S'}$ generated by $\{\alpha'(m+m') - \alpha'(m') - \alpha'(m)\}$

$\frac{V}{T}$ = submodule of $\frac{M}{T}$ generated by $\{\beta(m+m') - \beta(m') - \beta(m)\}$

and $\frac{V'}{T'}$ = submodule of $\frac{M}{T'}$ generated by $\{\beta'(m+m') - \beta'(m') - \beta'(m)\}$.

Now $[\alpha] = [\alpha']$ implies that there exists $T_1 \in \tau$ such that $T_1 \supseteq U+U' \supseteq S+S'$ and such that $\pi_{T_1 S}\alpha = \pi_{T_1 S'}\alpha'$. Also $[\beta] = [\beta']$ implies that there exists $T_2 \in \tau$ such that $T_2 \supseteq V+V' \supseteq T+T'$ and such that $\pi_{T_2 T}\beta = \pi_{T_2 T'}\beta'$.

Let $\frac{W}{C}$ = submodule of $\frac{M}{C}$ generated by $\{\tilde{\beta}\alpha(m+m') - \tilde{\beta}\alpha(m') - \tilde{\beta}\alpha(m)\}$ and $\frac{W'}{C'}$ = submodule of $\frac{M}{C'}$ generated by $\{\tilde{\beta}'\alpha'(m+m') - \tilde{\beta}'\alpha'(m') - \tilde{\beta}'\alpha'(m)\}$.

Then we have to show that there exists $T_3 \in \tau$ such that $T_3 \supseteq W+W' \supseteq C+C'$ and such that $\pi_{T_3 C}\tilde{\beta}\alpha = \pi_{T_3 C'}\tilde{\beta}'\alpha'$. For this let T_3/T_2 be the submodule of M/T_2 generated by $\pi_{T_2 T}\beta(T_1)$. Since $U \subseteq T_1$, $\pi_{T_2 T}\beta(U) \subseteq \pi_{T_2 T}\beta(T_1) \subseteq T_3/T_2$, so that $\beta(U) \subseteq T_3/T$. Hence $\tilde{\beta}(U/S) \subseteq T_3/C$ and $W/C \subseteq \tilde{\beta}(U/S)$, which implies that $W \subseteq T_3$. Similarly $W' \subseteq T_3$, hence $T_3 \supseteq W+W' \supseteq C+C'$. Since $\pi_{T_2 T}\beta$ is a homomorphism, $\pi_{T_2 T}\beta(m+m') - \pi_{T_2 T}\beta(m') - \pi_{T_2 T}\beta(m)$ is zero and hence $T_3 \in \tau$ by (iii).

Now $\pi_{T_3 C}\tilde{\beta}\alpha(m) = \pi_{T_3 C}\tilde{\beta}(x+S)$ where $\alpha(m) = x+S$

$= \pi_{T_3 C}\tilde{\beta}\nu_S(x)$ where $\nu_S : M \to \frac{M}{S}$ is the natural homomorphism.

$$= \pi_{T_3C}\, g\mu\beta(x)$$

$$= \pi_{T_3C}(y+C) \qquad \text{where} \quad \beta(x) = y+T.$$

$$= y + T_3$$

and $\quad \pi_{T_3C'}\tilde{\beta}'\alpha'(m) = \pi_{T_3C'}\,\tilde{\beta}'(x'+S') \quad \text{where } \alpha'(m) = x'+S'$

$$= \pi_{T_3C'}\,\tilde{\beta}'\,\nu_{S'}(x')$$

$$= \pi_{T_3C'}\, g'\mu'\beta'(x')$$

$$= \pi_{T_3C'}(y'+C') \quad \text{where} \quad \beta'(x') = y'+T'$$

$$= y'+T_3.$$

So that it remains to show that $y-y' \in T_3$. Now $\pi_{T_1S}\alpha(m) = \pi_{T_1S'}\alpha'(m)$ implies that $x + T_1 = x' + T_1$, so that $x - x' \in T_1$. Hence $\pi_{T_2T}\beta(x-x') \in \frac{T_3}{T_2}$, that is $\pi_{T_2T}\,\beta(x) - \pi_{T_2T}\,\beta(x') = (y+T_2) - (y'+T_2)$ $= (y-y') + T_2 \in \dfrac{T_3}{T_2}$. So that $y-y' \in T_3$, which implies that the multiplication is well defined.

It is easy to see that the multiplication is associative and that multiplication distributes over addition from the left.

Thus we have the following:

1.1. Theorem : P together with the above defined addition and multiplication is a left near ring with unity.

1.2 Definition : A submodule S of M is small if S+T = M implies T = M for all submodules T of M.

It is easy to see that the collection of small submodules satisfies the conditions (i) and (ii). To show that it also satisfies the condition (iii), consider $\alpha : M \rightarrow M/T$ as given in (iii), also let U/T be as given in (iii) where S and T are small submodules of M. Then we wish to show that U is small.

For this let B be a submodule of M such that U+B = M. Let

$$A = \{\sum_{i=1}^{n} (-m_i+x_i t_i+m_i) \mid \sum_{i=1}^{n}[(-m_i+T) + \alpha(x_i)t_i+(m_i+T)] \in \frac{T+B}{T} \text{ where } m_i, x_i \in M \text{ and either } t_i \text{ or } -t_i \in \text{ generating set } \bar{S} \text{ of } N\}$$

Case I : If A = M then for any $m \in M$, we have $m = -0+m+0 \in A$, which implies that $(-0+T) + \alpha(m) + (0+T) = \alpha(m) \in \frac{T+B}{T}$. Hence $\alpha(M) \subseteq \frac{T+B}{T}$.

For any $u \in U$, $u + T = \sum_{i=1}^{n} [(-m_i+T) + x_i t_i+(m_i+T)]$ where $t_i \in \bar{S} \cup (-\bar{S})$ and $x_i \in \alpha(S) \cup \{\alpha(m+m')-\alpha(m')-\alpha(m)\}$. Since $\alpha(M) \subseteq \frac{T+B}{T}$, $\alpha(S) \subseteq \frac{T+B}{T}$ and each $\alpha(m+m') - \alpha(m')-\alpha(m) \in \frac{T+B}{T}$, so that each x_i in the expression for $u + T$ is in $\frac{T+B}{T}$. Now since $\frac{T+B}{T}$ is a submodule of $\frac{M}{T}$, $u + T \in \frac{T+B}{T}$, which implies that $\frac{U}{T} \subseteq \frac{T+B}{T}$, so that $U \subseteq T+B$. Hence $U+B \subseteq T+B \subseteq U+B$, which implies

that $T+B = U+B = M$. This implies $B = M$ (since T is small), which shows that U is small.

Case II. If $A \neq M$ then $S+A \neq M$, this implies that there exists $m \in M$ such that $m \notin S + A$. Let $\alpha(m) = x+T$ for some $x = u+b \in M$.

$$\text{Then } \alpha(m) = (u+b)+T = (u+T)+(b+T) = \sum_{i=1}^{n} [(-m_i+T)+x_i t_i+(m_i+T)] + (b+T).$$

So that $- \sum_{i=1}^{n} [(-m_i+T)+x_i t_i + (m_i+T)] + \alpha(m) = b+T \in \frac{T+B}{T}$.

Now L.H.S. $= \{(-m_n+T)+\alpha(s_n)(-t_n)+(m_n+T)\} + [(-m_{n-1}+T)+ \{\alpha(m'+m'')-\alpha(m'')-\alpha(m')\}(-t_{n-1})+(m_{n-1}+T)]$

$$+ \ldots\ldots + [(-m_k+T) +\{\alpha(p_1+p_1')-\alpha(p_1')-\alpha(p_1)\}(-t_k)+(m_k+T)]+$$

$$+ \ldots\ldots + \alpha(m)$$

$$= \{(-m_n+T) + \alpha(s_n)(-t_n)+ (m_n+T)\} +$$

$$[(-m_{n-1}+T)+ \{\alpha(m'+m'')(-t_{n-1})- \alpha(m'')(-t_{n-1})- \alpha(m')(-t_{n-1})$$

$$+ (m_{n-1}+T)]+\ldots + [(-m_k+T)+ \{\alpha(p_1)t_k+ \alpha(p_1')t_k+ \alpha(p_1+p_1')(-t_k)\} + (m_k+ T)]$$

$$+ \ldots + \alpha(m) \quad \text{where } t_{n-1} \in -\overline{S} \text{ and } t_k \in \overline{S}$$

$$=\{(-m_n+T)+ \alpha(s_n)(-t_n)+ (m_n+T)\} + [\{(-m_{n-1}+T)+ \alpha(m'+m'')(-t_{n-1}) + (m_{n-1}+T)\}$$

$$+ \{(-m_{n-1}+T) + \alpha(m'')t_{n-1}+(m_{n-1}+T)\} + \{(-m_{n-1}+T)+ \alpha(m')t_{n-1}+(m_{n-1}+T)\}]$$

$+\ldots+ [\{(-m_k+T)+ \alpha(p_1)t_k+(m_k+T)\} + \{(-m_k+T)+ \alpha(p_1')t_k+(m_k+T)\}$

$+ \quad \{(-m_k+T) + \alpha(p_1+p_1')(-t_k) + (m_k+T)\}] + \ldots + \alpha(m)$

belongs to $\frac{T+B}{T}$ implies that $\{-m_n+s_n(-t_n)+ m_n\} + \{-m_{n-1}+(m'+m'')(-t_{n-1})+m_{n-1}\}$

$+ \{-m_{n-1}+m''t_{n-1}+m_{n-1}\}+\{-m_{n-1}+m't_{n-1}+ m_{n-1}\} + \ldots + \{-m_k+p_1t_k+m_k\} +$

$+ \{-m_k+ p_1't_k+ m_k\} + \{-m_k+(p_1+p_1')(-t_k)+m_k\} + \ldots + m \in A.$ Hence

$\{-m_n+ s_n(-t_n) + m_n\} + \{-m_{n-1}- m't_{n-1}- m''t_{n-1}+ m''t_{n-1}+ m't_{n-1}+ m_{n-1}\} +$

$+ \ldots + \{-m_k + p_1t_k + p_1't_k - p_1't_k-p_1t_k+ m_k\} + \ldots + m \in A.$ That is

$\{-m_n+ s_n(-t_n)+ m_n\} + \{0\} + \ldots + \{0\} + \ldots + m \in A.$ Hence $s + m \in A$ for some $s \in S$, which implies that $m \in S+A$, which is a contradiction.

When τ satisfies (i), (ii) and (iii), we say that M is τ-complemented if for each submodule S of M there exists a submodule T of M such that $S+T = M$ and $S \cap T \in \tau$.

1.3. Theorem : Assume τ satisfies (i), (ii) and (iii). If M is τ-complemented, then P is a regular near ring.

Proof : Let $[\alpha]$ be an arbitrary element of P where $\alpha \in P_T$. Suppose $\frac{U}{T}$ = submodule of $\frac{M}{T}$ generated by $\{\alpha(m+m') - \alpha(m')- \alpha(m)\}$.

Then $\beta : M \xrightarrow{\alpha} \frac{M}{T} \xrightarrow{\mu} \frac{M}{T}\Big|\frac{U}{T} \xrightarrow{g} \frac{M}{U}$ is an N-homomorphism,

where μ and g are the natu

We claim that $[\alpha] = [\beta]$. For this suppose $\frac{V}{U}$ is the submodule of $\frac{M}{U}$ generated by $\{\beta(m+m') - \beta(m') - \beta(m)\}$. Then $V = U$, since β is a homomorphism. By taking $T_1 = U$ we have $T_1 \in \tau$ such that $\pi_{T_1 T}\alpha = \pi_{T_1 U}\beta$, which implies that $\alpha \sim \beta$.

Now let $\beta(M) = \frac{A}{U}$ and $\overline{A}$ denote the submodule of M generated by A. Then there exists a submodule B of M such that $B + \overline{A} = M$ and $B \cap \overline{A} \in \tau$.

If $\pi : \frac{M}{U} \to \frac{M}{U + B \cap \overline{A}}$ denotes the natural homomorphism, then there exists a submodule C of M such that $C + \ker \pi\beta = M$ and $C \cap \ker \pi\beta \in \tau$. Now $\pi\beta$ induces an N-monomorphism $\tilde{\alpha}$ from $\frac{C}{C \cap \ker \pi\beta}$ to $\frac{M}{U + \overline{A} \cap B}$ such that $\tilde{\alpha}(c + C \cap \ker \pi\beta) = \pi\beta(c)$. Now for any $a \in A$,

$$\begin{aligned} a + (U + \overline{A} \cap B) &= \pi(a + U) \\ &= \pi\beta(x) \text{ for some } x = c+k \in M = C + \ker \pi\beta \\ &= \pi\beta(c) \\ &= \tilde{\alpha}(c + C \cap \ker \pi\beta). \end{aligned}$$

Since $\tilde{\alpha}$ is a monomorphism there is only one such element $c + C \cap \ker \pi\beta$ in $\frac{C}{C \cap \ker \pi\beta}$.

Now define $\theta : B + A \to \frac{M}{C \cap \ker \pi\beta}$ by

$\theta(b + a) = \tilde{\alpha}^{-1}(a + U + \overline{A} \cap B)$.

Now $b_1+a_1 = b_2+a_2$ implies that $(-b_2+b_1) = a_2-a_1 \in A \cap B \subseteq U + \bar{A} \cap B$. Hence $a_1+(U + \bar{A} \cap B) = a_2+(U + \bar{A} \cap B)$, which implies that

$\tilde{\alpha}^{-1}(a_1+U + \bar{A} \cap B) = \tilde{\alpha}^{-1}(a_2+U + \bar{A} \cap B)$. So that θ is well defined. It is easily seen that θ is a homomorphism.

Now define $\gamma : M \to \dfrac{M}{C \cap \ker \pi\beta}$ by

$\gamma(b + \sum_{i=1}^{n} (-m_i + a_i + m_i)) = \sum_{i=1}^{n} \theta(a_i)$. To see that γ is additive, take $b_1+ \sum_{i=1}^{n} (-m_i+a_i+m_i)$ and $b_2+ \sum_{i=1}^{k} (-m_i' + a_i' + m_i')$ in $M = B + \bar{A}$.

Then since

$$b_1+ \sum_{i=1}^{n} (-m_i+a_i + m_i) + b_2+ \sum_{i=1}^{k} (-m_i' + a_i' + m_i')$$

$$= b_1+ \{\sum_{i=1}^{n} (-m_i+a_i+m_i) +b_2- \sum_{i=1}^{n} (-m_i+a_i+m_i)\} + \sum_{i=1}^{n} (-m_i+a_i+m_i) + $$
$$+ \sum_{i=1}^{k} (-m_i'+a_i' + m_i')$$

$$= b + \sum_{i=1}^{n} (-m_i+ a_i+ m_i) + \sum_{i=1}^{k} (-m_i'+ a_i' + m_i') \in B + \bar{A}$$

where $b = b_1 + \{\sum_{i=1}^{n} (-m_i+ a_i+m_i) + b_2 - \sum_{i=1}^{n} (-m_i+ a_i+ m_i)\} \in B$, it is easy to see that γ is additive. Also $\gamma/A = \theta/A$ implies that $\gamma(0) = \theta(0) = 0$. Hence γ is well defined. It is easily seen that γ is a N-homomorphism.

We claim that $[\beta]\,[\gamma]\,[\beta] = [\beta]$. Now $[\beta]\,[\gamma]\,[\beta] =$ $[\beta]\,[\tilde{\gamma}\beta] = [\tilde{\beta}\cdot(\tilde{\gamma}\beta)]$ where $\tilde{\beta}$ is a homomorphism induced by $M \xrightarrow{\beta} \frac{M}{U} \xrightarrow{\mu_2} \frac{M}{U} \Big| \frac{W}{U} \xrightarrow{g_2} \frac{M}{W}$ and $\tilde{\gamma}$ is a homomorphism induced by $M \xrightarrow{\gamma} \frac{M}{C \cap \ker \pi\beta} \xrightarrow{\mu_1} M \,|\, C \cap \ker \pi\beta \,|\, V \,|\, C \cap \ker \pi\beta \xrightarrow{g_1} \frac{M}{V}$.

Also $\frac{V}{C \cap \ker \pi\beta}$ = submodule of $\frac{M}{C \cap \ker \pi\beta}$ generated by $\gamma(U)$ and $\frac{W}{U}$ = submodule of $\frac{M}{U}$ generated by $\beta(V)$.

Take $T_1 = W + \bar{A} \cap B \supseteq W$ then $T_1 \in \tau$ and

$$\pi_{T_1W}\,\tilde{\beta}\tilde{\gamma}\beta(m) = \pi_{T_1W}\,\tilde{\beta}\tilde{\gamma}(a + U), \text{where } \beta(m) = a + U.$$

$$= \pi_{T_1W}\,\tilde{\beta}\,g_1\mu_1\gamma(a)$$

$$= \pi_{T_1W}\,\tilde{\beta}g_1\,\mu_1\,\tilde{\alpha}^{-1}(a + U + \bar{A} \cap B)$$

$$= \pi_{T_1W}\,\tilde{\beta}g_1\,\mu_1(c + C \cap \ker \pi\beta\,)$$

$$= \pi_{T_1W}\,\tilde{\beta}(c + V)$$

$$= \pi_{T_1W}\,\tilde{\beta}\nu_V(c) \qquad \text{where } \nu_V\colon M \to \frac{M}{V}$$

$$= \pi_{T_1W}\,g_2\,\mu_2\,\beta(c)$$

$$= \pi_{T_1W}\,g_2\mu_2(d + U) \qquad \text{where } \beta(c) = d + U.$$

$$= \pi_{T_1W}(d + W)$$

$$= d + T_1\,.$$

Also $\pi_{T_1 U}\, \beta(m) = \pi_{T_1 U}(a + U) = a + T_1$.

Now $\tilde{\alpha}^{-1}(a + U + \bar{A} \cap B) = c + C \cap \ker \pi\beta$ implies that $a + (U + \bar{A} \cap B)$ $= \tilde{\alpha}(c + C \cap \ker \pi\beta) = \pi\beta(c) = \pi(d + U) = d + (U + \bar{A} \cap B)$. Hence $a - d$ $\varepsilon\ U + \bar{A} \cap B \subseteq W + \bar{A} \cap B = T_1$, that is $a + T_1 = d + T_1$. This implies that $[\beta][\gamma]\,[\beta] = [\beta]$. Hence $[\alpha]\ [\gamma]\ [\alpha] = [\alpha]$, so that P is a regular near ring.

We now consider a non-empty collection τ' of non-zero N-subgroups of M, satisfying

(a) If $S \ \varepsilon\ \tau'$ and $S \subseteq T$ then $T\ \varepsilon\ \tau'$

(b) If S and T belong to τ' then $S \cap T\ \varepsilon\ \tau'$

(c) If $S, T\ \varepsilon\ \tau'$ and $\alpha : S \to M$ is such that $(sr)\alpha = (s)\,\alpha r$

for all $s\ \varepsilon\ S$ and all distributive $r\ \varepsilon\ N$, then

$B = \{ \sum_{i=1}^{n} \pm x_i \ \mid \sum_{i=1}^{n} \pm (x_i)\alpha\ \varepsilon\ T,\ x_i\ \varepsilon\ S \}$ belongs to τ'

For each $T\ \varepsilon\ \tau'$ let $P'_T = \{\theta : T \to M \mid (tr)\theta = (t)\,\theta r$ for all $t\ \varepsilon\ T$ and all distributive $r\ \varepsilon\ N\}$.

Suppose $Y = \bigcup_{T\ \varepsilon\ \tau'} P'_T$. For each $\alpha\ \varepsilon\ P'_S$ and $\beta\ \varepsilon\ P'_T$ define $\alpha \sim \beta$ iff there exists $U\ \varepsilon\ \tau'$ such that $U \subseteq S \cap T$ and $\alpha|_U = \beta|_U$. It is easy to see that this relation is an equivalence relation. Denote by P' the set of equivalence classes of Y.

P' can be made into a near ring as follows :

is given by $(x)(\alpha+\beta) = (x)\alpha + (x)\beta$ for all $x \in S \cap T$, we define $[\alpha] + [\beta] = [\alpha + \beta]$. To see that the addition is well defined, consider $\alpha' \in P'_{S'}$ and $\beta' \in P'_{T'}$ such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Now $\alpha \sim \alpha'$ implies that there exists $U_1 \in \tau'$ such that $U_1 \subseteq S \cap S'$ and $\alpha|_{U_1} = \alpha'|_{U_1}$. Also $\beta \sim \beta'$ implies that there exists $U_2 \in \tau'$ such that $U_2 \subseteq T \cap T'$ and $\beta|_{U_2} = \beta'|_{U_2}$. Since $U_1 \cap U_2 \in \tau'$ and $\alpha + \beta$ coincides with $\alpha' + \beta'$ on $U_1 \cap U_2$ we have that the addition is well defined.

It is easy to see that P' is an additive group.

To define multiplication in P' consider $\alpha \in P'_S$ and $\beta \in P'_T$. Let $A = \{\sum_{i=1}^{n} \pm x_i \mid \sum_{i=1}^{n} \pm(x_i)\alpha \in T$ where $x_i \in S (1 \leq i \leq n)\}$. Then A is an N-subgroup of S which belongs to τ' (by condition (c)). Define $\phi : A \to T$ by $(\sum_{i=1}^{n} \pm x_i)\phi = \sum_{i=1}^{n} \pm (x_i)\alpha$. We note that ϕ is additive. To show that ϕ is well defined, consider $D = \{x \in S \mid (x)\alpha \in T\}$. Then D is contained in A and $\phi|_D = \alpha$. Since $0 \in D$ (for $(0)\alpha = 0 \in T$) we have $(0)\phi = 0$. Hence ϕ is well defined. We define $[\alpha][\beta] = [\phi \circ \beta]$

To see that the multiplication is well defined take $\alpha' \in P'_{S'}$ and $\beta' \in P'_{T'}$ such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Let

$A' = \{\sum_{i=1}^{m} \pm x_i \mid \sum_{i=1}^{m} \pm (x_i)\alpha' \in T'$ where $x_i \in S' (1 \leq i \leq m)\}$ and define $\phi' : A' \to T'$ by $(\sum_{i=1}^{m} \pm x_i)\phi' = \sum_{i=1}^{m} \pm (x_i)\alpha'$. Then $[\alpha'][\beta'] = [\phi' \circ \beta']$.

Now $\alpha \sim \alpha'$ implies that there exists $U \in \tau'$ such that $U \subseteq S \cap S'$ and $\alpha|_U = \alpha'|_U$. Also $\beta \sim \beta'$ implies that there exists $V \in \tau'$ such that $V \subseteq T \cap T'$ and $\beta|_V = \beta'|_V$. Let $B = \{\sum_{i=1}^{n} \pm x_i \mid x_i \in U \text{ and } \sum_{i=1}^{n} \pm (x_i)\alpha = \sum_{i=1}^{n} \pm (x_i)\alpha' \in V\}$. Then $B \in \tau'$ and $B \subseteq A \cap A'$. Moreover $\phi \circ \beta = \phi' \circ \beta'$ on B. Hence $[\phi \circ \beta] = [\phi' \circ \beta']$, which implies that the multiplication is well defined.

It can be easily worked out that P' is a non associative left near ring. Thus we have the following :

1.4. Theorem : P' is a non associative left near-ring.

1.5. Definition : An N-subgroup of M is large if it has non zero intersection with every non-zero N-subgroup of M.

It is easily seen that the collection of large N-subgroups of M satisfies the conditions (a) and (b). To see that it also satisfies (c), consider $\alpha : S \to M$, and B as given in (c) where S and T are large. We wish to show that B is large.

For this let U be a nonzero N-subgroup of M. Then $U \cap S \neq 0$.

Case I : If $(U \cap S)\alpha = 0$ then $(U \cap S)\alpha \subseteq T$, hence $0 \neq U \cap S \subseteq B$, that is $0 \neq U \cap S \subseteq U \cap B$. Hence B is large.

Case II : Suppose $(U \cap S)\alpha \neq 0$. Then there exists $0 \neq x \in U \cap S$ such that $(x)\alpha \neq 0$. Now $(x)\alpha N$ is a nonzero N-subgroup of M, hence $(x)\alpha N \cap T \neq 0$. So that there exists $n \in N$ such that $0 \neq (x)\alpha\, n \in T$,

that is $0 \neq (x)\alpha\, s_1 + (x)\alpha\, s_2 + \ldots + (x)\alpha\, s_k = (xs_1)\alpha + (xs_2)\alpha + + \ldots + (xs_k)\alpha \in T$. This implies $xs_1 + xs_2 + \ldots + xs_k \in B$ where $n = s_1+s_2+ \ldots + s_k$ for some s_i's $\in \overline{S}$ (generating set of N). Hence $xn \in B \cap U$. It remains to show that xn is non-zero.

For this consider $\phi : xN \rightarrow (x)\alpha N$ given by $(xr)\phi = (x)\alpha r$. We note that ϕ is additive. Also $\phi|_{x\overline{S}} = \alpha|_{x\overline{S}}$ and since $(0)\alpha = 0$ we have $(0)\phi = 0$. Hence ϕ is well defined.

Thus $(x)\alpha\, n \neq 0$ implies that xn is non zero, so that $B \cap U \neq 0$. Hence B is large.

If τ' satisfies (a), (b) and (c), we say that M is τ'-complemented if for each N-subgroup S of M there exists a submodule T of M such that $T + S \in \tau'$ and $T \cap S = 0$.

1.6. Theorem : Assume that τ' satisfies (a), (b) and (c). If M is τ'-complemented then P' is a regular near ring.

Proof : Let $[\beta]$ be an arbitrary element of P' where $\beta \in P'_U$. Let $\overline{K} = \{\sum_{i=1}^{n} \pm x_i \mid x_i \in U$ and $\sum_{i=1}^{n} \pm (x_i)\beta = 0\}$, then $\overline{K}$ is an N-subgroup of M contained in U. Hence there exists a submodule B of M such that $B + \overline{K} \in \tau'$ and $B \cap \overline{K} = 0$.

Let S be the N-subgroup of M generated by $(U \cap (B + \overline{K}))\beta$, then there exists a submodule C of M such that $C + S \in \tau'$ and $C \cap S = 0$. Define $\theta : (U \cap (B + \overline{K}))\beta \rightarrow M$ by $((b + \sum_{i=1}^{n} \pm k_i)\beta)\theta = b$. Now

$(b_1+\sum_{i=1}^{n} \pm k_i)\beta = (b_2+\sum_{i=1}^{m} \pm k_i')\beta$ implies that $(b_1+\sum_{i=1}^{n} \pm k_i)-$

$(b_2+\sum_{i=1}^{m} \pm k_i') = b_1 + \sum_{i=1}^{n} \pm k_i - \sum_{i=1}^{m} \pm k_i' - b_2 \in \overline{K}$. This implies

that $(b_1)\beta + \sum_{i=1}^{n} \pm (k_i)\beta - \sum_{i=1}^{m} \pm (k_i')\beta - (b_2)\beta = (b_1)\beta - (b_2)\beta = 0$,

so that $b_1-b_2 \in \overline{K} \cap B = 0$. Hence $b_1 = b_2$ which implies that θ is well defined.

Define $\gamma : C + S \to M$ by

$$(c + \sum_{i=1}^{n} \pm (b_i + \sum_{j=1}^{m_i} \pm k_{i_j})\beta)\gamma = \sum_{i=1}^{n} \pm b_i .$$

We note that γ is additive and $\gamma|_{(U \cap (B+\overline{K}))\beta} = \theta$.

Now $a = c + \sum_{i=1}^{n} \pm (b_i + \sum_{j=1}^{m_i} \pm k_{i_j})\beta = 0$ implies that

$\sum_{i=1}^{n} \pm (b_i + \sum_{j=1}^{m_i} \pm k_{i_j})\beta \in C \cap S = 0$. Hence $\sum_{i=1}^{n} \pm (b_i+ \sum_{j=1}^{m_i} \pm k_{i_j}) \in \overline{K}$,

which implies that $\sum_{i=1}^{n} ((b_i)\beta + \sum_{j=1}^{m_i} \pm (k_{i_j})\beta) = 0$, that is $\sum_{i=1}^{n} (b_i)\beta = 0$,

so that $\sum_{i=1}^{n} b_i \in B \cap \overline{K} = 0$. Hence $\sum_{i=1}^{n} b_i = 0$, showing that $(a)\gamma = 0$.

Hence γ is well defined. It is easily seen that γ is a N-homomorphism.

We claim that $[\beta] \{[\gamma] [\beta]\} = \{[\beta] [\gamma]\} [\beta] = [\beta]$.

We recall that $\beta : U \to M$ and $\gamma : C + S \to M$.

Let $A = \{\sum_{i=1}^{n} (\pm x_i) \mid x_i \in C + S, \ \sum_{i=1}^{n} \pm (x_i)\gamma \ \in \ U\}$ and define

$\phi : A \to U$ by $(\sum_{i=1}^{n} \pm x_i)\phi = \sum_{i=1}^{n} \pm (x_i)\gamma$ then $[\gamma]\,[\beta] = [\phi \circ \beta]$.

Now let $B = \{\sum_{i=1}^{n} (\pm x_i) \mid x_i \in U$ and $\sum_{i=1}^{n} \pm (x_i)\beta \ \in A\}$ and define

$\psi : B \to A$ by $(\sum_{i=1}^{n} \pm x_i)\psi = \sum_{i=1}^{n} \pm (x_i)\beta$, then

$[\beta]\,\{[\gamma]\,[\beta]\} = [\beta]\,[\phi \circ \beta] = [\psi \circ (\phi \circ \beta)]$.

Also let $A' = \{\sum_{i=1}^{n} \pm x_i \mid x_i \in U$ and $\sum_{i=1}^{n} \pm (x_i)\beta \ \in \ C + S\}$

and define $f : A' \to C + S$ by $(\sum_{i=1}^{n} \pm x_i)\, f = \sum_{i=1}^{n} \pm (x_i)\beta$, then

$[\beta]\,[\gamma] = [f \circ \gamma]$. Again let $B' = \{\sum_{i=1}^{n} \pm x_i \mid x_i \in A'$ and

$\sum_{i=1}^{n} \pm (x_i)\, f \circ \gamma \in U\}$

and define $g: B' \to M$ by $(\sum_{i=1}^{n} \pm x_i)\, g = \sum_{i=1}^{n} \pm (x_i) f \circ \gamma$. Then

$\{[\beta]\,[\gamma]\}\,[\beta] = [f \circ \gamma]\,[\beta] = [g \circ \beta]$.

It can be checked that $B \subseteq B'$ and $\psi_o(\phi \circ \beta) = g \circ \beta$ on B.

Hence $\{[\beta]\,[\gamma]\}\,[\beta] = [\beta]\,\{[\gamma]\,[\beta]\}$.

Now for all $(b + \sum_{i=1}^{n} \pm k_i) \in (B + \overline{K}) \cap U$ we have

$(b + \sum_{i=1}^{n} \pm k_i)\beta\,\gamma\,\beta = (b)\beta$.

Also $-b + (b + \sum_{i=1}^{n} \pm k_i) = \sum_{i=1}^{n} \pm k_i \in \overline{K}$, so we have

$-(b)\beta + (b + \sum_{i=1}^{n} \pm k_i)\beta = 0$ that is $(b)\beta = (b + \sum_{i=1}^{n} \pm k_i)\beta$.

Hence $(b + \sum_{i=1}^{n} \pm k_i)\beta\gamma\beta = (b + \sum_{i=1}^{n} \pm k_i)\beta$, which shows

that $[\beta]\,[\gamma]\,[\beta] = [\beta]$.

Hence P' is regular.

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